Wiener path integral based stochastic response determination of nonlinear systems with singular diffusion matrices

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Abstract

The recently developed Wiener Path Integral (WPI) technique for determining the joint response probability density function of stochastically excited nonlinear systems is extended herein to account for systems with singular diffusion matrices. Among others, indicative examples include partially forced systems, hysteresis modeling via additional auxiliary state equations, as well as certain electromechanical energy harvesting systems. Specifically, the governing equations of motion can be represented as an underdetermined set of stochastic differential equations (SDEs), coupled with a set of deterministic ordinary differential equations (ODEs) acting as constraints. Next, appropriately defining the Lagrangian function of the system leads to a constrained variational problem to be solved for the most probable path, and thus, for the system response PDF. Two numerical examples pertaining to both linear and nonlinear constraint equations are considered, whereas comparisons with Monte Carlo simulation data demonstrate a high degree of accuracy.

Keywords: singular diffusion matrix; path integral; constrained variational problem; stochastic dynamics; nonlinear system;

1. Introduction

Efficient analysis and design of engineering dynamical systems is often hindered by limitations pertaining to complex system nonlinearities and random excitations. Despite the versatility and simplicity of Monte Carlo simulation schemes as response analysis techniques, they become eventually prohibitive due to the associated high computational cost. As an alternative semi-analytical methodology, the recently developed Wiener path integral (WPI) technique appears promising in exhibiting both significant accuracy and computational efficiency. In particular, the WPI can address systems subject to non-white, non-Gaussian and non-stationary excitation processes [1], as well as endowed with fractional derivative elements [2]. Further, it's computational efficiency in addressing relatively high dimensional problems has been significantly improved recently [3],[4].

In this paper, the WPI technique is further generalized to cope with systems with singular diffusion matrices. Among others, indicative examples include partially forced systems, hysteresis modeling via additional auxiliary state equations, as well as certain electromechanical energy harvesting systems [5]. Specifically, the

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governing equations of motion can be represented as an underdetermined set of stochastic differential equations (SDEs), coupled with a set of deterministic ordinary differential equations (ODEs) acting as constraints. Next, appropriately defining the Lagrangian function of the system leads to a constrained variational problem to be solved for the most probable path, and thus, for the system response PDF. Two numerical examples pertaining to both linear and nonlinear constraint equations are considered, whereas comparisons with Monte Carlo simulation data demonstrate a high degree of accuracy.

2. Wiener path integral solution technique overview

2.1. Standard formulation

One of the recently developed promising techniques in stochastic engineering dynamics relates to the concept of the Wiener path integral (WPI) [6]. The technique exhibits not only relatively high accuracy in determining the joint response PDF, but also significant versatility as it can account for multi-degree-of-freedom systems with various nonlinearity types [7], systems with fractional derivative terms [2], as well as non-white, non-Gaussian and non-stationary excitation [1]. The essential aspects of the technique are delineated in the present section by considering the general class of n-dimensional randomly excited nonlinear structural/mechanical systems whose dynamics is described by

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t) \tag{1}$$

In Eq. (1), $\boldsymbol{x}$ is the system response, $\boldsymbol{M}$ is a mass matrix, $\boldsymbol{g}$ is a nonlinear function in general and $\boldsymbol{w}$ is a white noise stochastic excitation vector process with $E[\boldsymbol{w}(t_1)\boldsymbol{w}(t_2)] = \boldsymbol{D}\delta(t_2 - t_1)$; $\delta(.)$ denotes the Dirac delta function and $\boldsymbol{D}$ is a deterministic coefficient matrix given by

$$\boldsymbol{D} = \begin{bmatrix} 2\pi S_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 2\pi S_0 \end{bmatrix} \tag{2}$$

Next, relying on the mathematical framework of path integrals [8], the transition PDF $p(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i)$ can be written as [7]

$$p(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i) = \int\limits_{\mathcal{C}\{\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i; \boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f\}} W[\boldsymbol{x}(t)][\mathrm{d}\boldsymbol{x}(t)] \tag{3}$$

with $\{\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\}$ denoting the initial state and $\{\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f\}$ the final state, where $\boldsymbol{x}_i = \boldsymbol{x}(t_i)$, $\boldsymbol{x}_f = \boldsymbol{x}(t_f)$, $\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{x}}(t_i)$ and $\dot{\boldsymbol{x}}_f = \dot{\boldsymbol{x}}(t_f)$. Eq. (3) represents a functional integral over the space of all possible paths $\mathcal{C}\{\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i; \boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f\}$, $W[\boldsymbol{x}(t)]$ denotes the probability density functional of the stochastic process in the path space and $[\mathrm{d}\boldsymbol{x}(t)]$ is a functional measure. Further, the probability density functional for the stochastic vector process $\boldsymbol{x}(t)$ pertaining to the system of Eq. (1) is defined as (e.g., [7])

$$W[\boldsymbol{x}(t)] = \exp\left(- \int\limits_{t_i}^{t_f} \mathcal{L}\left(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}\right) \mathrm{d}t \right) \tag{4}$$

where $\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})$ denotes the Lagrangian function expressed as [7]

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \frac{1}{2}[\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})]^T \boldsymbol{D}^{-1}[\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})] \tag{5}$$

Note that Eq. (4) can be loosely interpreted as the probability assigned to each and every possible path starting from $\{\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\}$ and ending at $\{\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f\}$.

Clearly, the largest contribution to the functional integral of Eq. (3) comes from the trajectory $\boldsymbol{x}_c(t)$ for which the integral in the exponential of Eq. (4) (also known as stochastic action) becomes as small as possible; see [8] for instance. According to calculus of variations (e.g., [9],[10]) this trajectory $\boldsymbol{x}_c(t)$ with fixed endpoints satisfies the extremality condition

$$\delta \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) \mathrm{d}t = 0 \tag{6}$$

which leads to the Euler-Lagrange (E-L) equations

$$\frac{\partial \mathcal{L}}{\partial q_j} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} + \frac{\partial^2}{\partial t^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}_j} = 0, \quad j = 1, ..., n \tag{7}$$

with the set of boundary conditions

$$\begin{aligned} q_j(t_i) &= q_{j,i} \quad \dot{q}_j(t_i) = \dot{q}_{j,i} \\ q_j(t_f) &= q_{j,f} \quad \dot{q}_j(t_f) = \dot{q}_{j,f} \end{aligned} \qquad j = 1, ..., n \tag{8}$$

Next, solving equations (7)-(8) yields the n-dimensional most probable path, $\boldsymbol{x}_c(t)$, and thus, a single point of the system response transition PDF can be determined as [7]

$$p(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i) \approx C \ \exp\left(-\int_{t_i}^{t_f} L(\boldsymbol{x}_c, \dot{\boldsymbol{x}}_c, \ddot{\boldsymbol{x}}_c)\mathrm{d}t\right) \tag{9}$$

where C is merely a normalization constant. It can be readily seen by comparing equations (3) and (9) that in the approximation of Eq. (9) only one trajectory, i.e., the most probable path $\boldsymbol{x}_c(t)$, is considered in evaluating the path integral of Eq. (3). Regarding the degree of this approximation, direct comparisons of Eq. (9) with pertinent MCS data related to various engineering dynamical systems [7],[1],[2],[4], [5] have demonstrated satisfactory accuracy; see also [11].

Further, note that instead of directly solving the derived E-L equations (7)-(8), an alternative solution approach can be applied for determining the most probable path $\boldsymbol{x}_c(t)$. Specifically, since $\boldsymbol{x}_c$ is the solution of the variational problem

$$\text{minimize} \quad \mathcal{J}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) \mathrm{d}t, \tag{10}$$

i.e. it is is an extremum for the functional $\mathcal{J}$, calculus of variations rules suggest that a direct functional minimization formulation can be applied, which can be readily coupled with a standard Rayleigh-Ritz solution approach (see [12],[2],[13]). In this regard, $\boldsymbol{x}(t)$ is approximated by

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{\psi}(t) + \boldsymbol{R}\boldsymbol{h}(t) \approx \boldsymbol{x}(t). \tag{11}$$

The function $\boldsymbol{\psi}(t)$ is chosen so that it satisfies the boundary conditions, while the trial functions $\boldsymbol{h}(t) = [h_0(t), h_1(t), ..., h_{L-1}(t)]^T$ should vanish at the boundaries, i.e. $\boldsymbol{h}(t_i) = \boldsymbol{h}(t_f) = \mathbf{0}$. $\boldsymbol{R} \in \mathbb{R}^{n \times L}$ is a coefficient matrix, where L is the chosen number of trial functions considered. Clearly, there is a wide range of options for the choice of functions $\boldsymbol{\psi}$ and $\boldsymbol{h}$. In the ensuing analysis, the Hermite interpolating polynomials

$$\psi_j(t) = \sum_{k=0}^{3} \alpha_{j,k} t^k \tag{12}$$

are adopted, i.e., $\boldsymbol{\psi} = [\psi_1, \psi_2, ..., \psi_n]^T$, where the $n \times 4$ coefficients $\alpha_{j,k}$ are determined by the $n \times 4$ boundary conditions in Eq. (8). For the trial functions, the shifted Legendre polynomials given by the recursive formula

$$P_{p+1}(t) = \frac{2p+1}{p+1}\left(\frac{2t - t_i - t_f}{t_f - t_i}\right) P_p(t) - \frac{p}{p+1} P_{p-1}(t), \quad p = 1, 2, ... \tag{13}$$

are employed, which are orthogonal in the interval $[t_i, t_f]$, with $P_0(t) = 1$; and $P_1(t) = (2t - t_i - t_f)/(t_f - t_i)$. The trial functions take the form

$$h_l(t) = (t - t_i)^2 (t - t_f)^2 P_l(t). \tag{14}$$

A practical advantage of the Rayleigh-Ritz method is that the variational problem (functional minimization) degenerates to an ordinary minimization problem of a function that depends on a finite number of variables [10]. Specifically, the functional $\mathcal{J}$, dependent on the n functions $\boldsymbol{x}(t)$ (and their time derivatives), is replaced by the function $J(\boldsymbol{R})$, dependent on a finite number of $n \times L$ coefficients $\boldsymbol{R}$. Accordingly, the extremality condition in Eq. (6) is replaced by the optimality condition

$$\frac{\partial J(\boldsymbol{R})}{\partial \boldsymbol{R}} = \mathbf{0} \tag{15}$$

which represents essentially a set of nL nonlinear equations for the unknown coefficients (parameters) $\boldsymbol{R}$. Once solved numerically, the most probable path $\boldsymbol{x_c}$ is determined via Eq. (11).

2.2. Computational aspects

Considering fixed initial conditions $(\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i)$ (i.e., system initially at rest), both approaches, i.e. the E-L equations ((7)-(8)) and the Rayleigh-Ritz solution scheme, yield a single point of the joint response PDF via the solution of a deterministic boundary value problem (BVP). According to a brute-force implementation of the WPI technique, choosing a time instant t_f, an effective domain of values is considered for the joint response PDF $p(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i)$. Next, discretizing the effective domain using N points in each dimension, the joint response PDF values are obtained corresponding to the points of the mesh. Specifically, for an n-DOF system with $2n$ stochastic dimensions (n displacements and n velocities) the number of BVPs to be solved is N^{2n}. It is clear that the computational cost becomes prohibitive for relatively high-dimensional MDOF systems. However, efficient implementations, such as [3], based on the idea of employing a polynomial expansion for the joint response PDF, have been shown to significantly reduce the computational cost. A more recent improvement presented in [4], utilizes a compressive sensing approach to further reduce the computational cost of the WPI technique.

3. Extension of the Wiener path integral technique to account for singular diffusion matrices: A constrained variational problem

In the present work, attention is directed to a general class of systems with singular diffusion matrices, and in particular, to systems that can be cast in the form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \begin{bmatrix} \boldsymbol{w}(t) \\ \boldsymbol{0} \end{bmatrix} \tag{16}$$

where $\boldsymbol{M}$ is an $n \times n$, possibly singular mass matrix, and $\boldsymbol{g}$ is a nonlinear vector valued function. Clearly, comparing equations (1) and (16) , the $\boldsymbol{D}$ matrix corresponding to the right hand-side of Eq. (16) is singular, and thus, the Lagrangian of Eq. (5) cannot be defined in a straightforward manner.

In the following, singularity of $\boldsymbol{D}$ is treated by partitioning the system of Eq. (16) into two coupled subsystems. One that contains the equations corresponding to vector $\boldsymbol{w}$ on the RHS and one referring to the equations that correspond to the zero entries, i.e.,

$$\begin{bmatrix} \boldsymbol{M}_f\ddot{\boldsymbol{x}} + \boldsymbol{g}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}) \\ \boldsymbol{M}_u\ddot{\boldsymbol{x}} + \boldsymbol{g}_u(\boldsymbol{x}, \dot{\boldsymbol{x}}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}(t) \\ \boldsymbol{0} \end{bmatrix} \tag{17}$$

where subscripts f and u refer to "forced" and "unforced", respectively. Note that the upper subsystem, which will be referred to as the $f-$system, constitutes an undetermined system of $n - m$ SDEs and the lower subsystem, which will be referred to as the $u-$system, represents an undetermined system of m homogeneous ODEs. Therefore, it can be argued that the motion of the complete dynamical system in Eq. (16) is governed by the $f-$system of equations constrained, however, by the $u-$ system of equations.

In this regard, the solution to the considered problem is pursued by seeking for the solutions of the $f-$system that satisfy also the constraints of the $u-$ system. This leads to the formulation of a constrained variational problem for the determination of the most probable path $\boldsymbol{x}_c$, i.e.,

$$\text{minimize} \quad \mathcal{J}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \int_{t_i}^{t_f} \mathcal{L}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) \mathrm{d}t \tag{18}$$

$$\text{subject to} \quad \boldsymbol{\phi}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \boldsymbol{M}_u\ddot{\boldsymbol{x}} + \boldsymbol{g}_u(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{0} \tag{19}$$

where the Lagrangian function in Eq. (18) corresponds to the $f-$system only, and is given by

$$\mathcal{L}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \frac{1}{2}[\boldsymbol{M}_f\ddot{\boldsymbol{x}} + \boldsymbol{g}_f(\boldsymbol{x}, \dot{\boldsymbol{x}})]^T \boldsymbol{D}_f^{-1} [\boldsymbol{M}_f\ddot{\boldsymbol{x}} + \boldsymbol{g}_f(\boldsymbol{x}, \dot{\boldsymbol{x}})] \tag{20}$$

where $\boldsymbol{D}_f$ related to the f-system is the nonsingular square submatrix of $\boldsymbol{D}$.

Constrained variational problems of the form of Eqs. (18)-(19) can be solved by using the general Lagrange multiplier approach of [14] and [15]. This leads to an unconstrained variational problem by considering the auxiliary Lagrangian function

$$\mathcal{L}^*(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \mathcal{L}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) + \boldsymbol{\lambda}(t)\boldsymbol{\phi}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) \tag{21}$$

This yields a system of n Euler-Lagrange equations similar to the ones in Eq. (7)-(8), to be solved together with the m constraint function in Eq. (19) for the n unknown functions $\boldsymbol{x}(t)$ and the m unknown Lagrange multiplier functions $\boldsymbol{\lambda}(t)$. In practice, however, the reformulation of this complex system of $n+m$ equations to an equivalent first order, required by most numerical BVP solvers, requires multiple time differentiations of the constraint functions. As a result, the time-derivatives of the constraints are fulfilled, but not the constraints themselves. This is a common limitation in several numerical methods for BVPs, as highlighted in [16]. Therefore, in the present work, attention is directed to a Rayleigh-Ritz solution approach for the determination of the most probable path.

In this regard, the polynomial expansion $\hat{\boldsymbol{x}}(t) = \boldsymbol{\psi}(t) + \boldsymbol{R}\boldsymbol{h}(t)$ (see Eq. (11) and the explanation below) is adopted for the response vector $\boldsymbol{x}(t)$ that reduces the functional $\mathcal{J}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})$ to a function $J(\boldsymbol{r}) := \mathcal{J}_f(\hat{\boldsymbol{x}}, \dot{\hat{\boldsymbol{x}}}, \ddot{\hat{\boldsymbol{x}}})$, where $\boldsymbol{r} \in \mathbb{R}^p$, is the vectorized form of the expansion parameter matrix $\boldsymbol{R} \in \mathbb{R}^{n\times L}$ and $p = nL$. In addition, the functions $\hat{\boldsymbol{\phi}}(\boldsymbol{r}, t) := \boldsymbol{\phi}(\hat{\boldsymbol{x}}, \dot{\hat{\boldsymbol{x}}}, \ddot{\hat{\boldsymbol{x}}})$ are defined, and the constraints in Eq. (19) are replaced by

$$\hat{\boldsymbol{\phi}}(\boldsymbol{r}, t) = \boldsymbol{0} \tag{22}$$

The adoption of the Rayleigh-Ritz solution approach allows for the reduction of the constrained variational problem in Eq. (18)-(19) to an ordinary constrained optimization problem, and facilitates its numerical treatment.

3.1. Linear constraints

In the special case that the constraint functions in Eq. (19) are linear in $\boldsymbol{x}$ and its time derivatives, i.e. $\boldsymbol{g}_u(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x}$, where $\boldsymbol{C}$ and $\boldsymbol{K}$ are square matrices, a considerably efficient solution scheme can be pursued by restricting the optimization within the space of solutions of Eq. (22) via a nullspace approach. Specifically, linearity of the constraint equations ensures that $\hat{\phi}(\boldsymbol{r}, t)$ is a vector of m polynomial functions, each of degree $L+4$ in t (see equations (11) - (14)), with coefficients linear in the nL unknown expansion parameters $\boldsymbol{r}$. Setting these polynomial coefficients equal to zero, yields a set of $m(L+4)$ linear equations with $p = nL$ unknown variables. Next, these equations are cast as a linear system of the form

$$\boldsymbol{A}\boldsymbol{r} = \boldsymbol{b} \tag{23}$$

where $\boldsymbol{A} \in \mathbb{R}^{s\times p}$, $\boldsymbol{b} \in \mathbb{R}^s$ and $s = m(L+4)$. Of course, for any well-posed constrained optimization problem, the number of independent constraints is smaller than the dimension of $\boldsymbol{x}$. For the herein concerned problem, this yields $m(L+4) < p$, which provides the lower bound $L > \frac{4m}{n-m}$ for the number L of Legendre polynomials used in the polynomial expansion. The system in Eq. (23) is underdetermined, while $\boldsymbol{A}$ may not have full row rank, i.e., $r_A \leq s$.

The corresponding constrained optimization problem

$$\underset{\boldsymbol{r}\in\mathbb{R}^p}{\arg\min}\, J(\boldsymbol{r}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{r} = \boldsymbol{b} \tag{24}$$

of dimension p is recast next into a lower dimensional unconstrained problem of dimension $p - r_A$ as

$$\underset{\boldsymbol{v}\in\mathbb{R}^{p-r_A}}{\arg\min}\, J(\boldsymbol{S}\boldsymbol{v} + \boldsymbol{r}_p) \tag{25}$$

where $\boldsymbol{S} = [\boldsymbol{s}_1\ \boldsymbol{s}_2\ \ldots\ \boldsymbol{s}_{p-r_A}] \in \mathbb{R}^{p\times(p-r_A)}$ is a basis for the *nullspace* of $\boldsymbol{A}$ [17] and $\boldsymbol{r}_p$ is any particular solution of Eq. (23) [17], [18]; see also [19]. Note that the minimizer $\boldsymbol{r}^*$ of Eq. (24) can be obtained by the minimizer $\boldsymbol{v}^*$ of Eq. (25) as $\boldsymbol{r}^* = \boldsymbol{S}\boldsymbol{v}^* + \boldsymbol{r}_p$.

3.2. Nonlinear constraints

In the more general case of nonlinear constraints, it is possible to formulate an optimization problem with nonlinear equality constraints and utilize a suitable solution method, e.g. the augmented Lagrangian method (ALM). More specifically, a necessary and sufficient condition for Eq. (22) to hold is

$$\boldsymbol{\xi}(\boldsymbol{r}) := \sqrt{\int_{t_i}^{t_f} \hat{\boldsymbol{\phi}}^2(\boldsymbol{r}, t)\mathrm{d}t} = \mathbf{0} \tag{26}$$

and the corresponding optimization problem can be formulated as

$$\underset{\boldsymbol{r}\in\mathbb{R}^p}{\arg\min}\, J(\boldsymbol{r}) \quad \text{subject to} \quad \boldsymbol{\xi}(\boldsymbol{r}) = \mathbf{0} \tag{27}$$

In general, optimization problems of this form are among the most challenging ones [20], with the *augmented Lagrangian method* (ALM) being one of the typically used solution approaches. The ALM approximates the solution of the problem in Eq. (27) by successively minimizing the *augmented Lagrangian function*

$$L_A(\boldsymbol{r}, \boldsymbol{\lambda}; \mu) = J(\boldsymbol{r}) - \sum_{i=1}^{m} \lambda_i \xi_i(\boldsymbol{r}) + \frac{\mu}{2}\sum_{i=1}^{m} \xi_i^2(\boldsymbol{r}) \tag{28}$$

for an increasing sequence of penalty factors μ. Therefore, a sequence of unconstrained subproblems is formulated, where the solution of the previous problem is used as the initial guess for the next one, i.e.,

$$\boldsymbol{r}^{k+1} = \underset{\substack{\boldsymbol{r}\in\mathbb{R}^p \\ \boldsymbol{r}_{init}=\boldsymbol{r}^k}}{\arg\min}\, L_A(\boldsymbol{r}, \boldsymbol{\lambda}^k; \mu^k) \tag{29}$$

where the Lagrange multiplier vector $\boldsymbol{\lambda}$ at each step is given by the explicit estimate

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k - \mu^k \boldsymbol{\xi}(\boldsymbol{r}^k) \tag{30}$$

and $\boldsymbol{r}_{init}$ denotes the initial guess for the solution of the corresponding optimization problem.

The ALM has been shown to improve the ill-posedeness of the well known *quadratic penalty method* (QPM), as it can approximate the solution of the original problem even with moderate values of the penalty factor μ [20]. Also, note that the augmented Lagrangian function in Eq. (28) can be derived as the dual of the corresponding *quadratic penalty function* of the QPM, as shown in [21]. Finally, for the solution of the unconstrained subproblems as well as for the problem in Eq. (25), a standard quasi-Newton method is utilized, in conjunction with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) formula [22], for the explicit approximation of the Hessian matrix.

4. Numerical Examples

To demonstrate the reliability of the proposed technique for determining the response PDF of stochastically excited MDOF systems with singular diffusion matrices, two indicative examples are considered in this section. The first example pertains to a 2-DOF oscillator, where only the first DOF is stochastically excited. Further, nonlinearities relate only to the first DOF as well, so that the second equation is linear, and can be treated via the linear constraints approach of Sec. 3.1. The second example refers to the same system as above, but the constraint equation is nonlinear as well and is treated via the approach of Sec. 3.2.

4.1. Partially forced 2-DOF oscillator

The following partially forced 2-DOF oscillator is considered in the present section, with linear and with nonlinear constraints, i.e.,

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \begin{bmatrix} w(t) \\ 0 \end{bmatrix} \tag{31}$$

where

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{32}$$

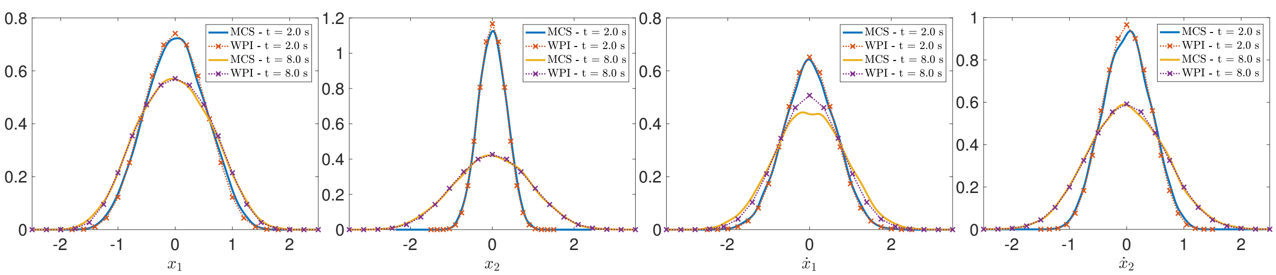
and $S_0 = 0.1$.

4.1.1. Linear constraints

First, a version of the 2-DOF oscillator of Eq. (31) with stiffness and damping nonlinearities in the first equation and linear second equation is considered; thus, yielding linear constraints in the proposed computational framework. In particular, the nonlinear function $\boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})$ takes the form

$$\boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \epsilon \begin{bmatrix} c_{11}\dot{x}_1^3 + k_{11}x_1^3 \\ 0 \end{bmatrix} \tag{33}$$

where x_1 is the first component of the response vector $\boldsymbol{x}$, c_{11} and k_{11} are the upper left elements of matrices $\boldsymbol{C}$ and $\boldsymbol{K}$ respectively, and the magnitude of the nonlinearity ϵ is taken equal to 0.5.



(a) $p(x_1)$ (b) $p(x_2)$ (c) $p(\dot{x}_1)$ (d) $p(\dot{x}_2)$

Figure 1: Marginal response PDFs of a partially forced 2-DOF nonlinear oscillator with linear constraints.

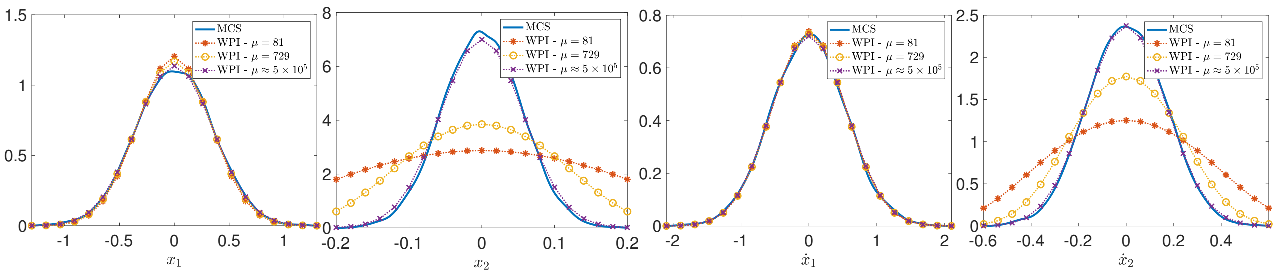
The WPI technique in conjunction with the methodology described in Sec. 3.1 is utilized next, and the joint response PDF $p(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is calculated for two time instants $t = 2$ s and $t = 8$ s. The corresponding marginal response PDFs are presented in Fig. 1, and compared with pertinent Monte Carlo simulation (MCS) results (10000 realizations), demonstrating a high degree of accuracy. A standard fourth-order Runge-Kutta numerical integration scheme is employed for solving the governing equations of motion within the MCS context.

4.1.2. Nonlinear constraints

Next, a second version of the 2-DOF oscillator in Eq. (31) with stiffness nonlinearities in both equations is considered; thus, yielding nonlinear constraints in the proposed computational framework. In this case, the nonlinear function $\boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})$ takes the form

$$\boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \epsilon \begin{bmatrix} k_{11} x_1^3 \\ k_{22} x_2^3 \end{bmatrix} \tag{34}$$

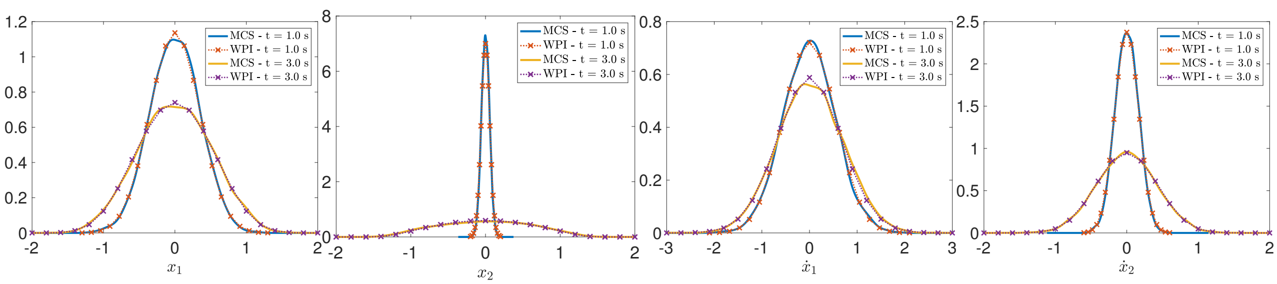
where x_1 and x_2 are the first and second components of the response vector $\boldsymbol{x}$, k_{11} and k_{22} are the upper left and lower right elements of matrix $\boldsymbol{K}$ respectively, and the nonlinearity magnitude ϵ is taken equal to 0.5.



(a) $p(x_1)$ (b) $p(x_2)$ (c) $p(\dot{x}_1)$ (d) $p(\dot{x}_2)$

Figure 2: Marginal response PDFs of a partially forced 2-DOF nonlinear oscillator with nonlinear constraints at $t = 1$ s for increasing values of the penalty factor μ.

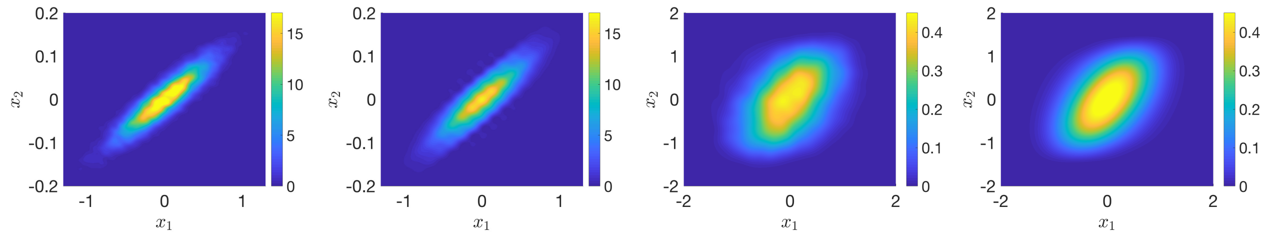
The WPI technique in conjunction with the methodology described in Sec. 3.2 is utilized next. In this context, the joint response PDF $p(\boldsymbol{x}, \dot{\boldsymbol{x}})$ at time $t = 1$ s is obtained using the ALM, where the augmented Lagrangian function is being sequentially minimized for the increasing sequence of penalty factors $\mu = 1$, 9, 81, 729, 6561, 59049, 531441. After appropriate integration of the joint PDF, the corresponding marginal response PDFs are obtained for three values of μ and presented in Fig. 2. A comparison with pertinent Monte Carlo simulation (MCS) results (10000 realizations) demonstrates the convergence of the marginal PDFs to the MCS estimates for increasing μ.



(a) $p(x_1)$ (b) $p(x_2)$ (c) $p(\dot{x}_1)$ (d) $p(\dot{x}_2)$

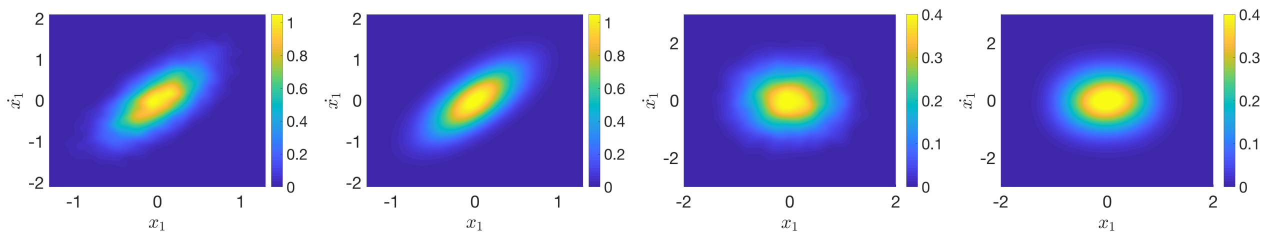
Figure 3: Marginal response PDFs of a partially forced 2-DOF nonlinear oscillator with nonlinear constraints at $t = 1$ s for increasing values of the penalty factor μ.

Moreover, the marginal response PDFs of all the response quantities for sufficiently large values of μ for two time instants $t = 1$ s and $t = 3$ s, are shown in Fig. 3. In addition, the joint response PDFs $p(x_1, x_2)$, $p(x_1, \dot{x_1})$ and $p(x_2, \dot{x_2})$ are shown in Figures 4, 5 and 6, respectively, for the two time instants $t = 1$ s and $t = 3$ s as well. In a similar manner as before, comparisons with MCS data demonstrate a high degree of accuracy.



(a) MCS - $t = 1$ s (b) WPI - $t = 1$ s (c) MCS - $t = 3$ s (d) WPI - $t = 3$ s

Figure 4: Joint response PDF $p(x_1, x_2)$ of a partially forced 2-DOF nonlinear oscillator with nonlinear constraints at $t = 1$ s and $t = 3$ s.



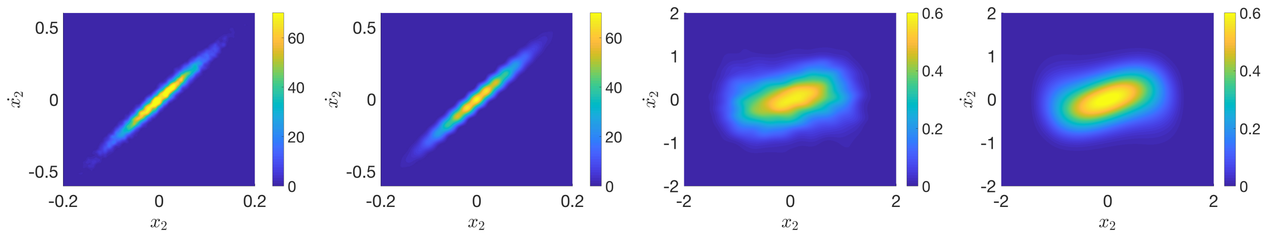
(a) MCS - $t = 1$ s (b) WPI - $t = 1$ s (c) MCS - $t = 3$ s (d) WPI - $t = 3$ s

Figure 5: Joint response PDF $p(x_1, \dot{x}_1)$ of a partially forced 2-DOF nonlinear oscillator with nonlinear constraints at $t = 1$ s and $t = 3$ s.

5. Concluding remarks

A methodology based on the WPI technique has been developed for determining the nonstationary joint response PDF of a class of nonlinear dynamical systems with singular diffusion matrices. In this regard, the WPI technique [7],[3],[6] has been extended herein to account for systems that can be represented, generally, as an underdetermined system of SDEs coupled with a set of ODEs. Interpreting the latter as constraint equations leads to a constrained variational problem to be solved for the most probable path. To this aim, a direct functional minimization formulation has been applied, coupled with a standard Rayleigh-Ritz solution approach (see [5],[12]). This has reduced the constrained variational problem to an ordinary constrained optimization problem. It has been found that a nullspace solution approach is computationally efficient for cases of linear constraints, whereas the augmented Lagrangian method (ALM) has performed satisfactorily for cases of nonlinear constraints.

In the present work, the reliability of the methodology is demonstrated by two numerical examples; that is, a 2-DOF oscillator with Duffing nonlinearities and linear constraint equation, and a 2-DOF oscillator with Duffing nonlinearities in the constraint equation as well. Comparisons with pertinent MCS data demonstrate a high degree of accuracy.



(a) MCS - $t = 1$ s (b) WPI - $t = 1$ s (c) MCS - $t = 3$ s (d) WPI - $t = 3$ s

Figure 6: Joint response PDF $p(x_2, \dot{x_2})$ of a partially forced 2-DOF nonlinear oscillator with nonlinear constraints at $t = 1$ s and $t = 3$ s.

References

[1] A. F. Psaros, O. Brudastova, G. Malara, I. A. Kougioumtzoglou, Wiener path integral based response determination of nonlinear systems subject to non-white, non-gaussian, and non-stationary stochastic excitation, Journal of Sound and Vibration 433 (2018) 314–333.

[2] A. Di Matteo, I. A. Kougioumtzoglou, A. Pirrotta, P. D. Spanos, M. Di Paola, Stochastic response determination of nonlinear oscillators with fractional derivatives elements via the Wiener Path Integral, Probabilistic Engineering Mechanics 38 (2014) 127–135.

[3] I. A. Kougioumtzoglou, A. Di Matteo, P. D. Spanos, A. Pirrotta, M. Di Paola, An efficient Wiener path integral technique formulation for stochastic response determination of nonlinear mdof systems, Journal of Applied Mechanics 82 (10) (2015) 101005.

[4] A. F. Psaros, I. A. Kougioumtzoglou, I. Petromichelakis, Sparse representations and compressive sampling for enhancing the computational efficiency of the Wiener path integral technique, Mechanical Systems and Signal Processing 111 (2018) 87–101.

[5] I. Petromichelakis, A. F. Psaros, I. A. Kougioumtzoglou, Stochastic response determination and optimization of a class of nonlinear electromechanical energy harvesters: A Wiener path integral approach, Probabilistic Engineering Mechanics 53 (2018) 116–125.

[6] I. Kougioumtzoglou, P. Spanos, An analytical Wiener path integral technique for non-stationary response determination of nonlinear oscillators, Probabilistic Engineering Mechanics 28 (2012) 125–131.

[7] I. A. Kougioumtzoglou, P. D. Spanos, Nonstationary stochastic response determination of nonlinear systems: A Wiener path integral formalism, Journal of Engineering Mechanics 140 (9) (2014) 04014064.

[8] M. Chaichian, A. Demichev, Path Integrals in Physics: Stochastic Processes and Quantum Mechanics, Institute of Physics Publishing, Bristol, U.K., 2001.

[9] G. M. Ewing, Calculus of Variations with Applications, Dover, New York, 1985.

[10] L. D. Elsgolc, Calculus of variations, Dover Publications, Mineola, N.Y, 2007.

[11] A. T. Meimaris, I. A. Kougioumtzoglou, A. A. Pantelous, A closed form approximation and error quantification for the response transition probability density function of a class of stochastic differential equations, Probabilistic Engineering Mechanics 54 (2018) 87–94.

[12] I. A. Kougioumtzoglou, A Wiener Path Integral Solution Treatment and Effective Material Properties of a Class of One-Dimensional Stochastic Mechanics Problems, Journal of Engineering Mechanics 143 (6) (2017) 04017014.

[13] O. C. Zienkiewicz, Finite elements and approximation, Dover Publications, Mineola, N.Y, 2006.

[14] C. Lanczos, The Variational Principles of Mechanics (Dover Books on Physics), Dover Publications, Mineola, NY, 1986.

[15] M. Giaquinta, S. Hildebrandt, Calculus of Variations I (Grundlehren der mathematischen Wissenschaften) (Vol 1), Springer, Berlin, 2006.

[16] C. W. Gear, B. Leimkuhler, G. K. Gupta, Automatic integration of euler-lagrange equations with constraints, Journal of Computational and Applied Mathematics 12 (1985) 77–90.

[17] G. Strang, Linear Algebra and Its Applications, 4th Edition, Thomson Higher Education, Belmont, CA, 2006.

[18] G. E. Shilov, Linear Algebra (Dover Books on Mathematics), Dover Publications, Mineola, NY, 1977.

[19] E. N. Antoniou, A. A. Pantelous, I. A. Kougioumtzoglou, A. Pirrotta, Response determination of linear dynamical systems with singular matrices: A polynomial matrix theory approach, Applied Mathematical Modelling 42 (2017) 423–440.

[20] J. Nocedal, S. Wright, Numerical Optimization (Springer Series in Operations Research and Financial Engineering), Springer, New York, 2006.

[21] D. P. Bertsekas, Constrained optimization and Lagrange multiplier methods, Athena Scientific, Nashua, NH, 1982.

[22] D. Goldfarb, A family of variable-metric methods derived by variational means, Mathematics of computation 24 (109) (1970) 23–26.