8th Computational Stochastic Mechanics Conference (CMS8)
Paros, Greece
June 9-13, 2018

Wiener path integral based stochastic response determination of nonlinear systems with singular diffusion matrices

Ioannis Petromichelakis, Apostolos F. Psaros and Ioannis A. Kougioumtzoglou

Dept. of Civil Engineering & Engineering Mechanics
Columbia University, USA

Outline

- Introduction
- Wiener path integral (WPI) technique
  - Standard formulation
  - Modification
- E-L equations and Lagrange Multipliers
- Rayleigh-Ritz and Constrained Optimization
  - Linear constraints
  - Nonlinear constraints
- Conclusions

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Introduction

- Engineering Stochastic Dynamics
- Wiener Path Integral (WPI) techniques $\longrightarrow$ transition PDF
- Theory of Stochastic Differential Equations (SDEs) $\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x}, t) + \tilde{\boldsymbol{B}}(\boldsymbol{x}, t)\boldsymbol{\eta}(t)$
- Mechanical oscillators under white noise excitation: 2nd order SDEs

Chaichian and Demichev (2001) Path integrals in physics. Vol. 1. CRC Press

Singular diffusion matrix

- Mechanical oscillators under non-white excitation: higher order SDEs

Psaros, Brudastova, Malara & Kougioumtzoglou, J Sound & Vibration (Under Review)

Introduction

- A wide class of stochastic dynamics problems can be modeled as:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \begin{bmatrix} \boldsymbol{w}(t) \\ \mathbf{0} \end{bmatrix}$$

- $\boldsymbol{w}(t)$: White noise vector process

  Cases:
  - Filtered white noise excitation processes
  - Nonlinear vibratory energy harvesters
  - Partially (stochastically) forced structures
  - Hysteretic systems, e.g. Bouc-Wen oscillator

- Lead to singular diffusion matrices

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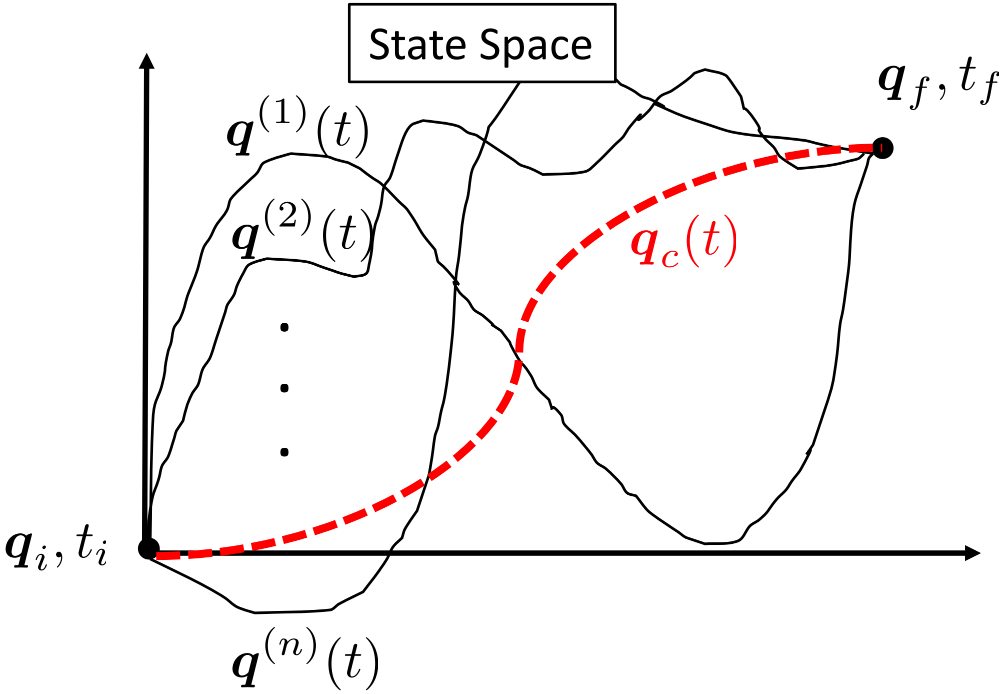
WPI technique - Standard formulation

- Wiener path integral (WPI) ⟶ Wiener (1921), Feynman (1948)
- Transition probability density $p(\boldsymbol{q}_f, t_f | \boldsymbol{q}_i, t_i)$

$$= \int_{\mathcal{C}\{\boldsymbol{q}_i, t_i; \boldsymbol{q}_f, t_f\}} W[\boldsymbol{q}(t)][\mathrm{d}\boldsymbol{q}(t)] = \int_{\mathcal{C}\{\boldsymbol{q}_i, t_i; \boldsymbol{q}_f, t_f\}} \Phi \exp\left(-\int_{t_i}^{t_f} \underbrace{\mathcal{L}}_{\text{Lagrangian functional}}(\boldsymbol{q})\,\mathrm{d}t\right)[\mathrm{d}\boldsymbol{q}(t)]$$

$$\approx \Phi \exp\left(-\int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{q}_c)\,\mathrm{d}t\right)$$



- Determine $\boldsymbol{q}_c$ by solving:

Variational problem

minimize $\mathcal{J}(\boldsymbol{q}) = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{q})\,\mathrm{d}t$

→ Euler-Lagrange equations

→ Rayleigh-Ritz direct method

WPI technique - Standard formulation

$$\mathcal{J}\left(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}\right) = \int_{t_i}^{t_f} \mathcal{L}\left(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}\right) \mathrm{d}t$$

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t) \longrightarrow \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \cancel{\frac{1}{2}[\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})]^T \boldsymbol{B}^{-1}[\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})]}$$

1. From Calculus of Variations

   Euler-Lagrange equations

   extremality condition: $\delta\mathcal{J}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = 0 \longrightarrow$

$$\frac{\partial \mathcal{L}}{\partial x_j} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}_j} + \frac{\partial^2}{\partial t^2}\frac{\partial \mathcal{L}}{\partial \ddot{x}_j} = 0, \quad j = 1, \ldots, n$$

2. The **Rayleigh-Ritz direct method**

   $\boldsymbol{x}(t) \approx \boldsymbol{\psi}(t) + \boldsymbol{c}^T\boldsymbol{h}(t)$ polynomial basis expansion of the response

   $n \times L$ coefficient matrix

   - Then the functional $\mathcal{J}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})$ becomes a function of $\boldsymbol{c} \longrightarrow J(\boldsymbol{c})$
   - Enables the utilization of **optimization theory** and **algorithms**

- If the diffusion matrix is singular $\longrightarrow$ $\boldsymbol{B}$ is singular

Treatment of diffusion matrix singularity

- Separation of the governing equations into two underdetermined systems

$$\begin{bmatrix} \boldsymbol{M_f}\ddot{\boldsymbol{x}} + \boldsymbol{g_f}(\boldsymbol{x},\dot{\boldsymbol{x}}) \\ \boldsymbol{M_u}\ddot{\boldsymbol{x}} + \boldsymbol{g_u}(\boldsymbol{x},\dot{\boldsymbol{x}}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}(t) \\ \boldsymbol{0} \end{bmatrix}$$

→ SDEs → $n - m$ system equations

→ Homogeneous ODEs → m constraints

- The Lagrangian $\mathcal{L}_f$ of the system equations is written as:

$$\mathcal{L}_f(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \frac{1}{2}[\boldsymbol{M_f}\ddot{\boldsymbol{x}} + \boldsymbol{g_f}(\boldsymbol{x},\dot{\boldsymbol{x}})]^T \boldsymbol{B_f}^{-1}[\boldsymbol{M_f}\ddot{\boldsymbol{x}} + \boldsymbol{g_f}(\boldsymbol{x},\dot{\boldsymbol{x}})]$$

$\boldsymbol{B_f}$: **non-singular** square submatrix of $\boldsymbol{B}$

Constrained variational problem

minimize $\mathcal{J}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \int_{t_i}^{t_f} \mathcal{L}_f(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}})\,\mathrm{d}t$

subject to $\boldsymbol{M_u}\ddot{\boldsymbol{x}} + \boldsymbol{g_u}(\boldsymbol{x},\dot{\boldsymbol{x}}) = \boldsymbol{0}$

→ Two solution approaches

1. Euler-Lagrange equations
2. **Rayleigh-Ritz direct method**

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EL equations and Lagrange multipliers

- From Calculus of Variations

  Unconstrained variational problem

  minimize $\mathcal{J}^*(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \int_{t_i}^{t_f} \mathcal{L}^*(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}})\,\mathrm{d}t$

  Lagrange multiplier vector function

  where

  $$\mathcal{L}^*(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \mathcal{L}_f(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) + \boldsymbol{\lambda}(t)^T(\boldsymbol{M_u}\ddot{\boldsymbol{x}} + \boldsymbol{g_u}(\boldsymbol{x},\dot{\boldsymbol{x}}))$$

- The most probable path $\boldsymbol{x}_c(t)$ is the solution of the system:

$$\left.\begin{aligned} &\frac{\partial \mathcal{L}^*}{\partial x_j} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}^*}{\partial \dot{x}_j} + \frac{\partial^2}{\partial t^2}\frac{\partial \mathcal{L}^*}{\partial \ddot{x}_j} = 0, \quad j = 1, \ldots, n \\ &\boldsymbol{M_u}\ddot{\boldsymbol{x}} + \boldsymbol{g_u}(\boldsymbol{x},\dot{\boldsymbol{x}}) = \boldsymbol{0} \end{aligned}\right\}$$

n E-L equations + **m** constraints

- Reduction of these high order ODEs to first order ⟵ Numerical methods

  requires multiple differentiations of the constraints ⟶ Computational complications

  Gear, Leimkuhler, and Gupta (1985) *J Comp. & Applied Math.*

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Rayleigh-Ritz method and Constrained Optimization

- Polynomial basis expansion $\boldsymbol{x}(t) \approx \boldsymbol{\psi}(t) + \boldsymbol{c}^T \boldsymbol{h}(t)$

  functional $\mathcal{J}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \int_{t_i}^{t_f} \mathcal{L}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}})\,\mathrm{d}t \longrightarrow$ <u>nonlinear function</u> $J(\boldsymbol{c})$

- **Linear constraints** $\boldsymbol{M}_{\boldsymbol{u}}\ddot{\boldsymbol{x}} + \boldsymbol{g}_{\boldsymbol{u}}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{M}_{\boldsymbol{u}}\ddot{\boldsymbol{x}} + \boldsymbol{C}_{\boldsymbol{u}}\dot{\boldsymbol{x}} + \boldsymbol{K}_{\boldsymbol{u}}\boldsymbol{x} = \boldsymbol{0} \longrightarrow \phi(\boldsymbol{c}, t) = 0$
  - functions $\boldsymbol{\phi}$ are polynomials in t with coefficients **linear** in $\boldsymbol{c}$
  - equate all polynomial coefficients to zero $\rightarrow$ $\boldsymbol{A}\boldsymbol{c} - \boldsymbol{b} = \boldsymbol{0}$
  - minimize $J(\boldsymbol{c})$ subject to $\boldsymbol{A}\boldsymbol{c} - \boldsymbol{b} = \boldsymbol{0}$
    $\boldsymbol{c} \in \mathbb{R}^{L \times n}$ $\longrightarrow$ Nonlinear optimization problem with <u>linear equality constraints</u>

- Efficient solution using nullspace of $\boldsymbol{A}$

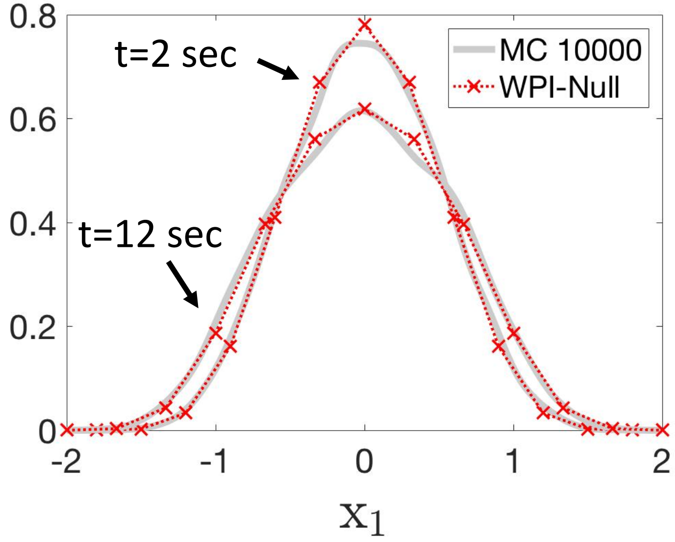
  restrict $\boldsymbol{c}$ to the subspace of $\mathbb{R}^{L \times n}$ where $\boldsymbol{A}\boldsymbol{c} - \boldsymbol{b} = \boldsymbol{0}$ is always satified

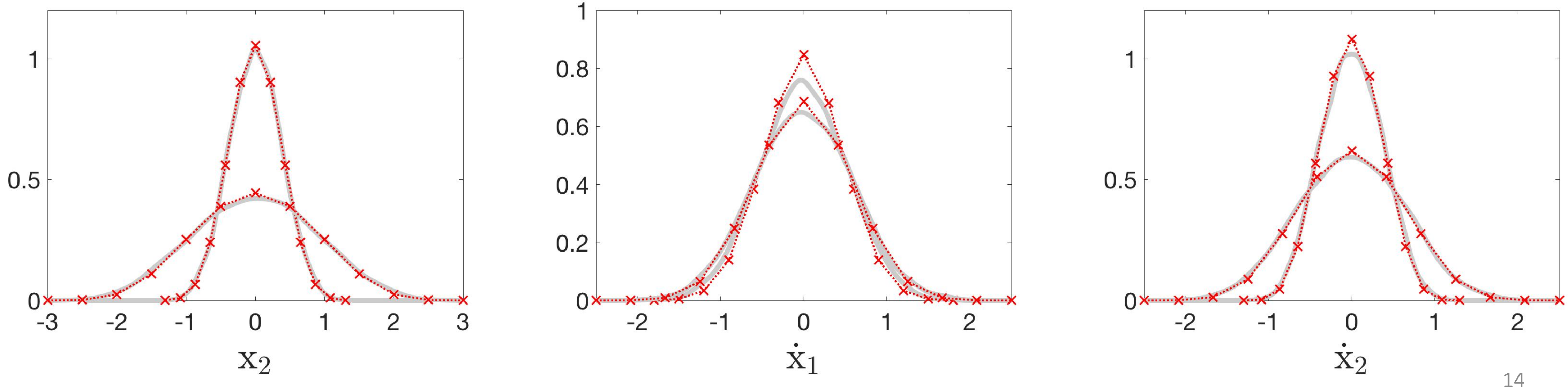
1st Example: Partially forced 2 DOF oscillator

$$\boldsymbol{M}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\boldsymbol{C}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\boldsymbol{K}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+0.5\begin{bmatrix}c_{11}\dot{x}_1^3+k_{11}x_1^3\\ 0\end{bmatrix}=\begin{bmatrix}w(t)\\ 0\end{bmatrix}$$

nonlinear eq. of motion

linear constraint





Rayleigh-Ritz method and Constrained Optimization

- Polynomial basis expansion $\boldsymbol{x}(t) \approx \boldsymbol{\psi}(t) + \boldsymbol{c}^T \boldsymbol{h}(t)$

  functional $\mathcal{J}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \int_{t_i}^{t_f} \mathcal{L}_f(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) \, \mathrm{d}t \longrightarrow$ nonlinear function $J(\boldsymbol{c})$

- **Nonlinear constraints** $\boldsymbol{M}_{\boldsymbol{u}} \ddot{\boldsymbol{x}} + \boldsymbol{g}_{\boldsymbol{u}}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{\phi}(\boldsymbol{c}, t) = \boldsymbol{0} \iff \boldsymbol{\xi}(\boldsymbol{c}) = \sqrt{\int_{t_i}^{t_f} \boldsymbol{\phi}^2(\boldsymbol{c}, t) \mathrm{d}t} = \boldsymbol{0}$

  - $\boxed{\text{minimize}_{\boldsymbol{c} \in \mathbb{R}^{L \times n}} \; J(\boldsymbol{c}) \; \text{subject to} \; \boldsymbol{\xi}(\boldsymbol{c}) = \boldsymbol{0}}$ $\longrightarrow$ Nonlinear optimization problem with **nonlinear** equality constraints

- Augmented Lagrangian method

$$L_A(\boldsymbol{c}, \boldsymbol{\lambda}; \mu) = J(\boldsymbol{c}) \cancel{- \sum_{i=1}^{m} \lambda_i \xi_i(\boldsymbol{c})} + \frac{\mu}{2} \sum_{i=1}^{m} \xi_i^2(\boldsymbol{c})$$

Dual of the Quadradic Penalty Method (QPM)

Bertsekas (1985) *Constrained Opt. and Lagrange Multiplier Methods*

for $k = 0, 1, \ldots$ and increasing μ^k

$$\boldsymbol{c}^{k+1} = \arg\min L_A(\boldsymbol{c}^k, \boldsymbol{\lambda}^k; \mu^k)$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k - \mu^k \boldsymbol{\xi}(\boldsymbol{c}^k)$$

Computational improvement over the QPM

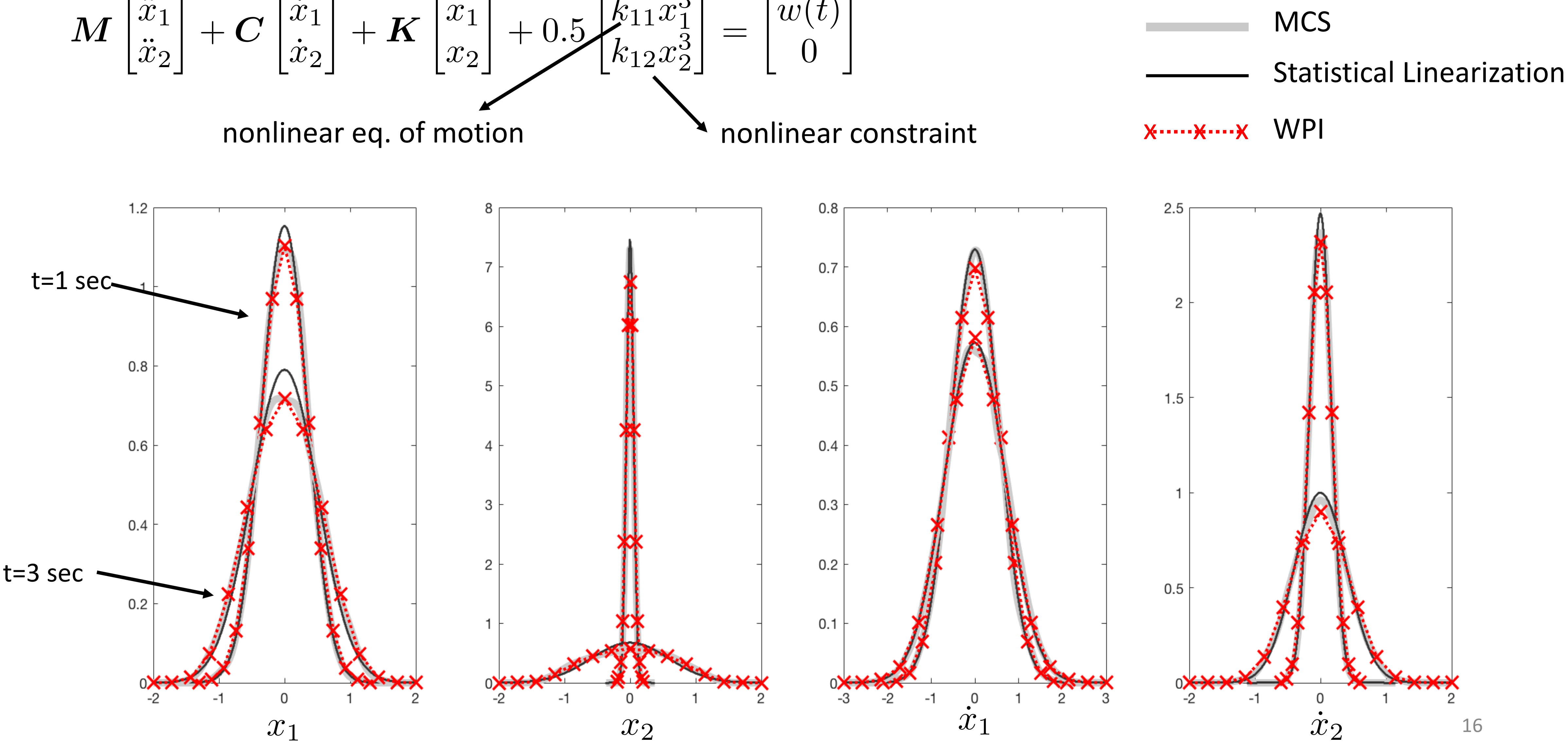
Nocedal, Wright (2006) *Numerical Optimization*

2nd Example: Partially forced 2 DOF oscillator

$$\boldsymbol{M}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\boldsymbol{C}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\boldsymbol{K}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+0.5\begin{bmatrix}k_{11}x_1^3\\ k_{12}x_2^3\end{bmatrix}=\begin{bmatrix}w(t)\\ 0\end{bmatrix}$$

nonlinear eq. of motion

nonlinear constraint

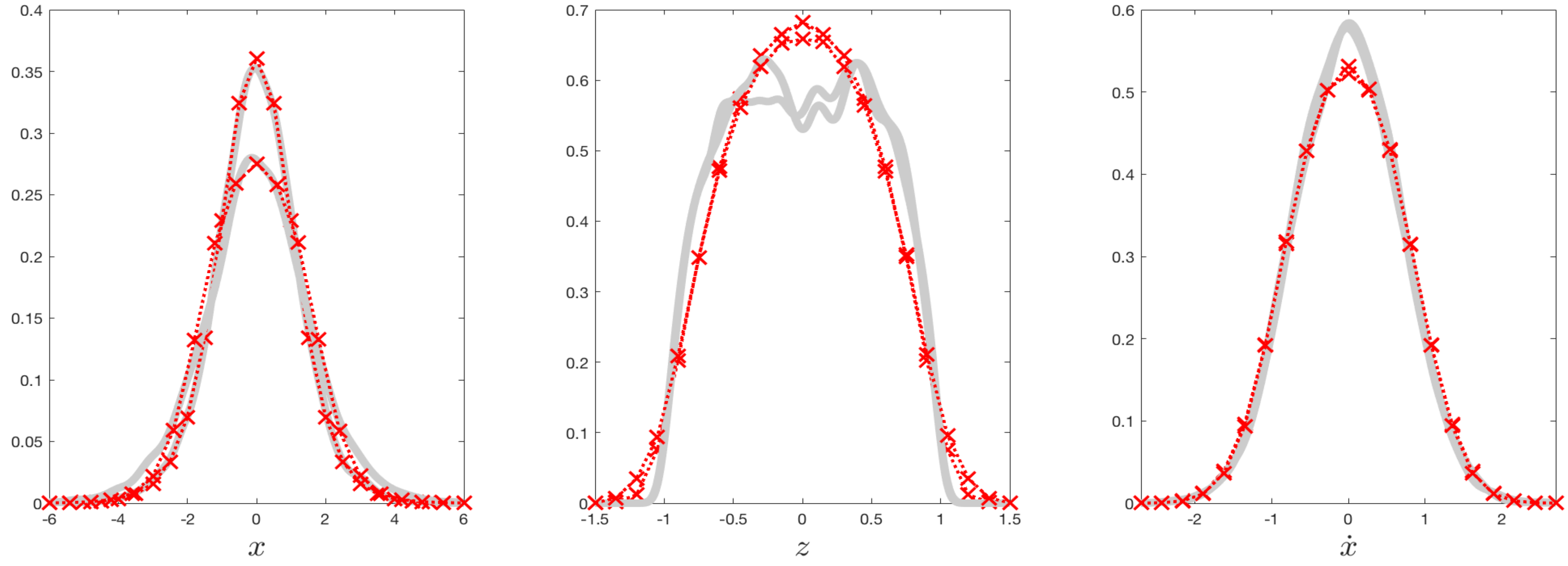


3rd Example: Bouc-Wen hysteretic oscillator

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \alpha\omega_0^2 x + (1-\alpha)\omega_0^2 z = w(t) \quad \longrightarrow \quad \text{system equation}$$

$$\dot{z} + \gamma|\dot{x}|z|z|^{\nu-1} + \beta\dot{x}|z|^{\nu} - A\dot{x} = 0 \quad \longrightarrow \quad \text{constraint}$$

- Marginal response PDFs for ν = 1 at t = 10 and 20 sec and increasing penalty factor μ

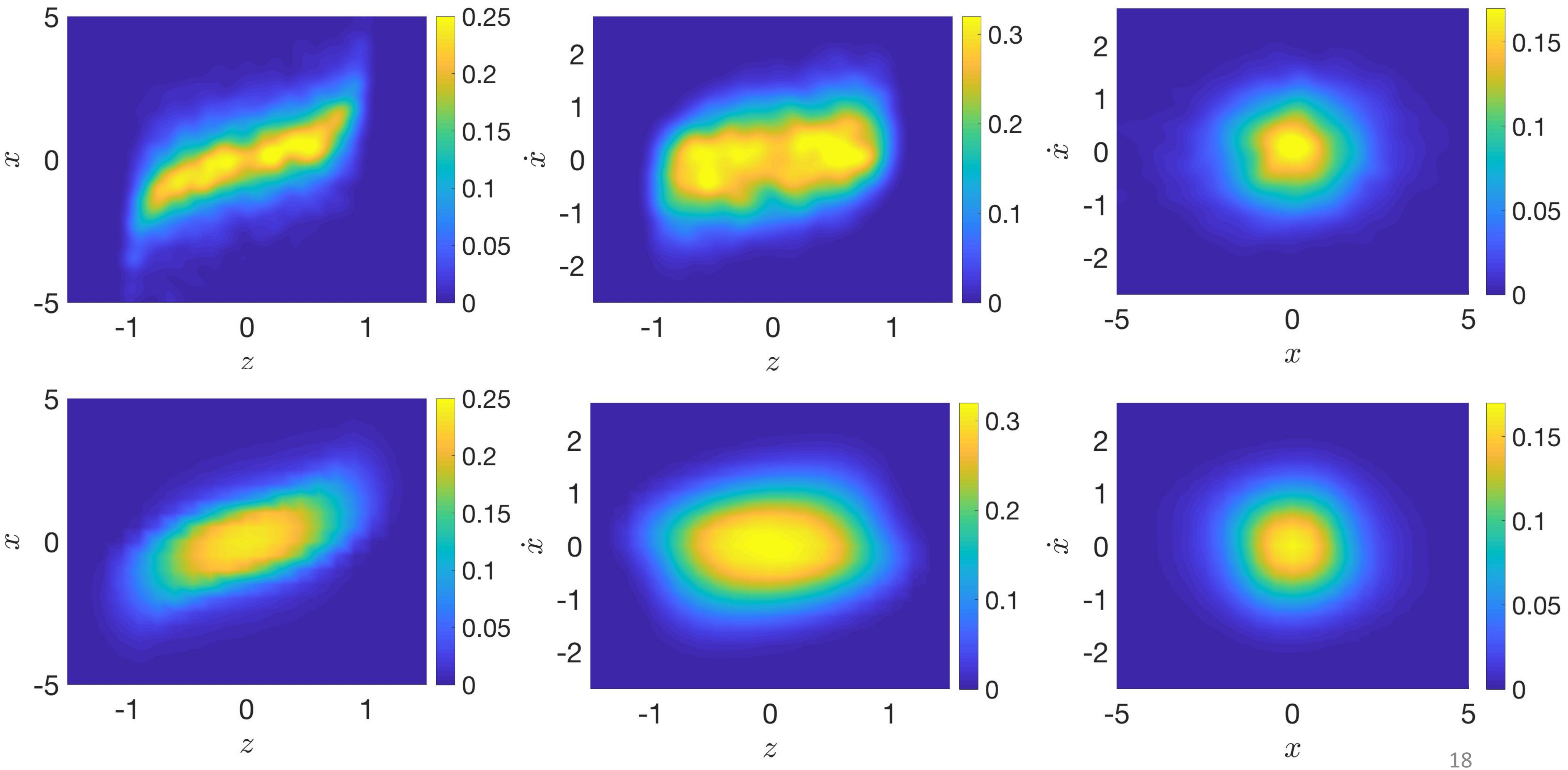


3rd Example: Bouc-Wen hysteretic oscillator

- 2D joint response PDFs for ν = 1 at t = 15 sec

MC 10000 realizations

WPI – Augmented Lagrangian method



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Conclusions

- Modification of the standard WPI technique to account for systems with singular diffusion matrices
- Separation of the system equations into two underdetermined sub-systems and formulation of a constrained variational problem
  - E-L equations and Lagrange multipliers
    - Theoretically rigorous – Calculus of Variations
    - Computational limitations
  - Rayleigh-Ritz and constrained optimization
    - Approximate but more versatile
    - Linear constraints: Nullspace approach (very efficient)
    - Nonlinear constraints: Augmented Lagrangian method
- Examples: Partially forced MDOF oscillators with linear and nonlinear constraints, Bouc-Wen hysteretic oscillator

Thank you!

Appendix: Nonlinear energy harvester example

$$\ddot{x} + 2\zeta\dot{x} + x + \lambda x^2 + \delta x^3 + \kappa^2 y = w(t)$$
$$\dot{y} + \alpha y - \dot{x} = 0$$

- Solution: EL equation and Lagrange multipliers
- marginal response PDFs