Influence of the boundaries in imaging for damage localization in 1D domains

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Detection and Localization of Damage

- Usually based on response recordings at a number of sensors to monitor structural integrity [1]
- Detection : comparison of recordings to a reference (undamaged) state
- Localization : Inverse Problem usually ill-posed
- Solution : *Time-Reversal* (TR) computational tool introduced by Fink et. al. [2]
- Achieves refocusing of the wave on the source
- Sending back the recorded signals but reversed in time
- Two step approach
  - Forward step
  - Backward step

1. GE Stavroulakis, (2000) Inverse and crack identification problems in engineering mechanics
2. Fink et. al., (2000) Time-reversed acoustics

Time Reversal and applications

- TR is a physical process
- It exploits the time reversibility (based on spatial reciprocity and time reversal invariance) of linear wave equations
- Robust and Simple technique for source localization
- Has been applied in Acoustics [3], Elastodynamics [4], Electromagnetism, Hydrodynamics etc.
- Finds several applications in medicine, telecommunications, underwater acoustics, seismology, engineering structures, etc.
- TR can be used for scatterer localization
- The fundamental idea of TR can be exploited to develop different imaging techniques

3. L Borcea, G Papanicolaou, C Tsogka and J Berryman, (2002) Imaging and time reversal in random media
4. D Givoli, (2014) Time Reversal as a Computational Tool in Acoustics and Elastodynamics

In the present work

- Acoustic medium in an 1D bounded domain
- Description of the numerical implementation of TR
- Utilization of the Green's function of the Helmholtz equation to apply imaging techniques based on TR
- Investigation of the influence of the boundaries using modal expansion of the Green's function
- Investigation of the influence of the total experiment time T
- Demonstration of numerical examples
- Proposition of techniques to improve the quality of the image and the SNR
- Indicative demonstration of a 2D example

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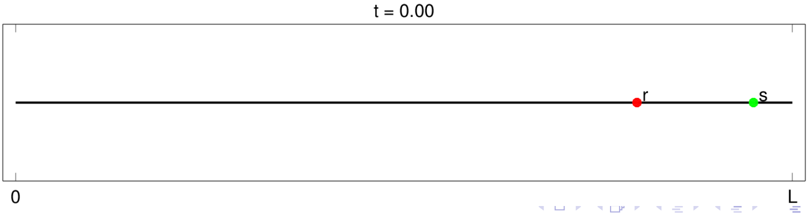
Forward step

- The domain contains one source at x_s and one receiver at x_r
- Simulated Numerically using a finite element method
- Wave propagation model

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = f(t)\delta(x - x_s)$$

- Homogeneous Dirichlet boundary conditions and zero initial conditions
- Excitation function $f(t)$ is a Ricker pulse centered at a known t_0
- The response $p(x_r, t)$ is being recorded during total time T



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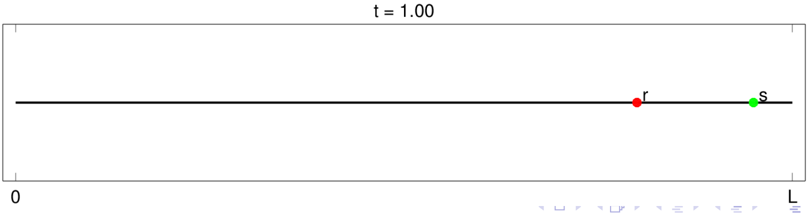

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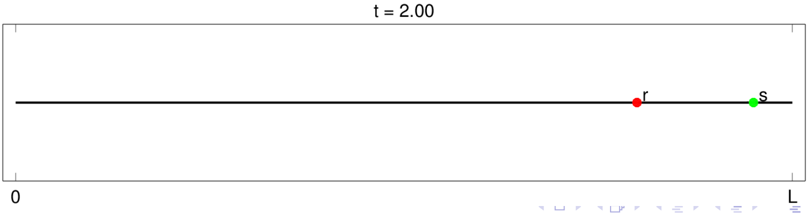

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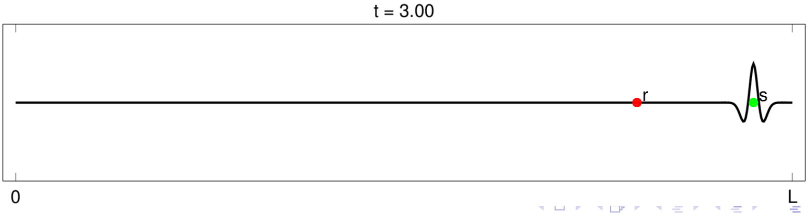

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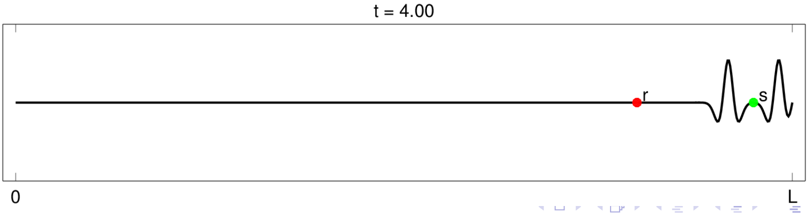

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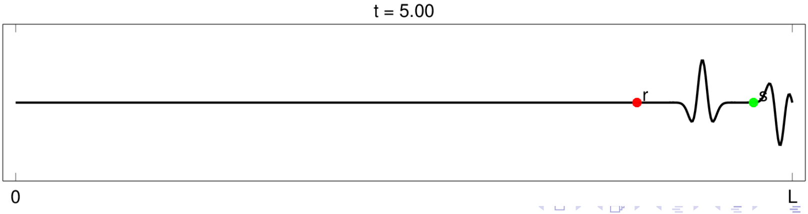

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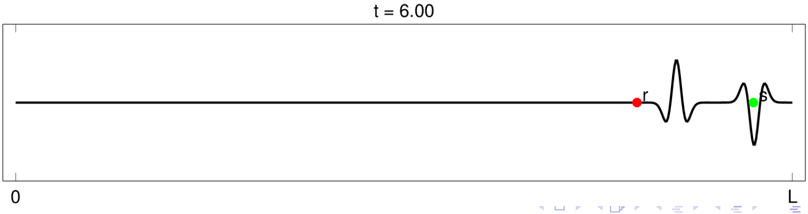

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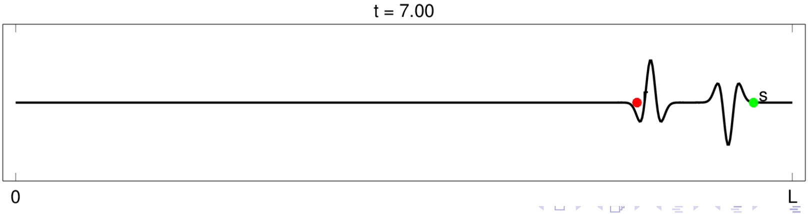

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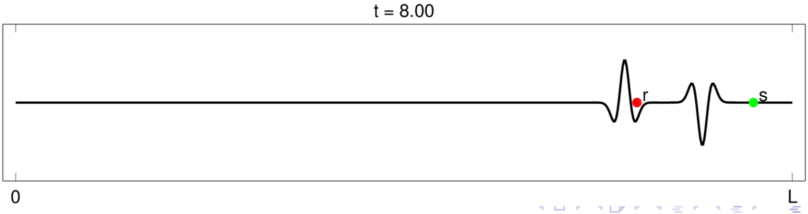

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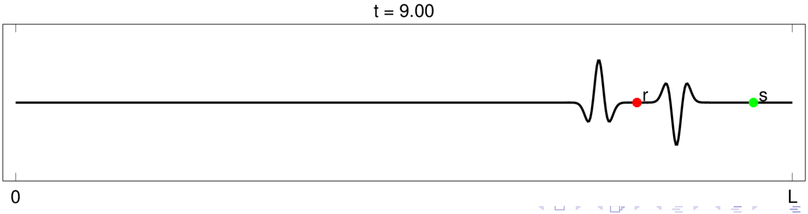

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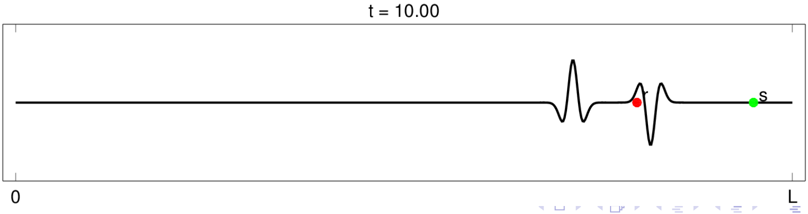

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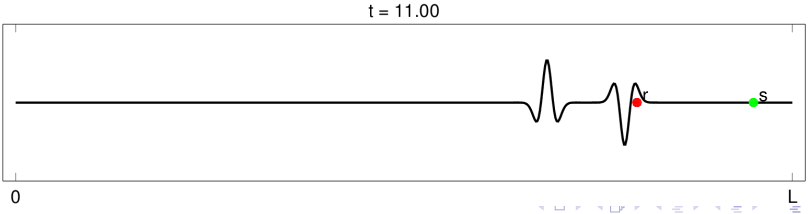

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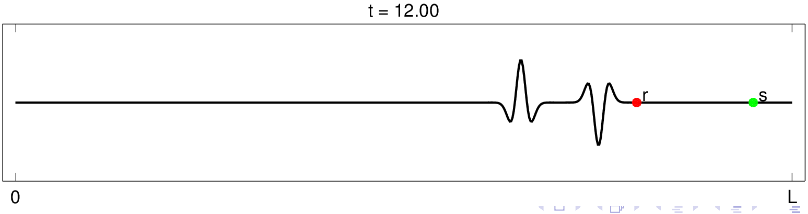

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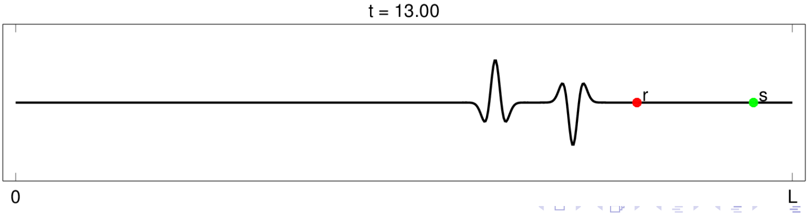

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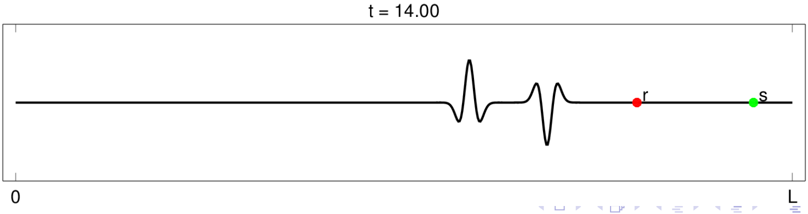

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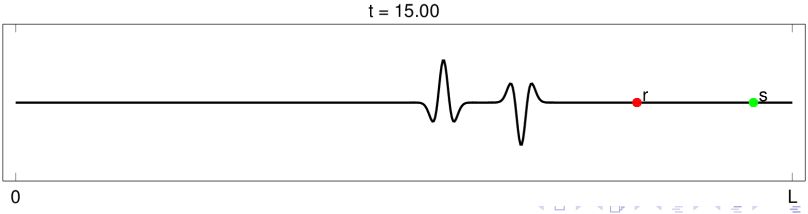

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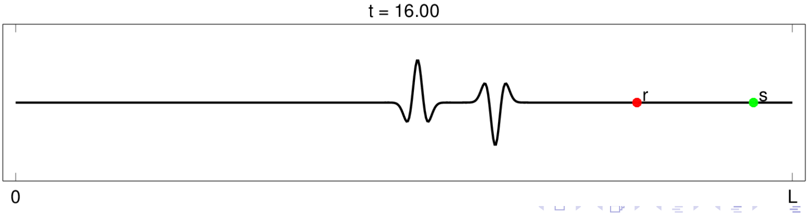

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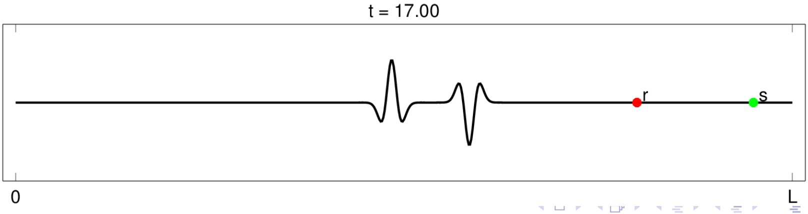

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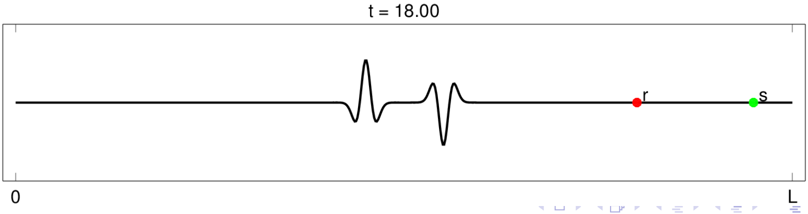

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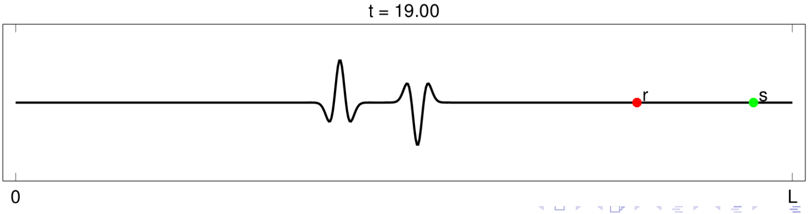

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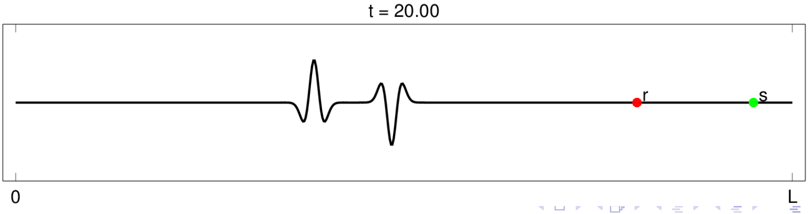

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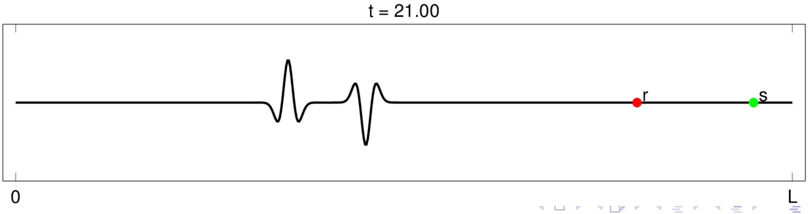

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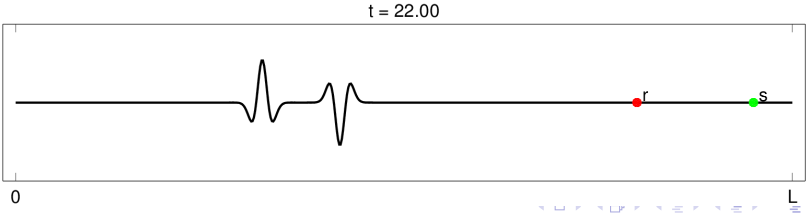

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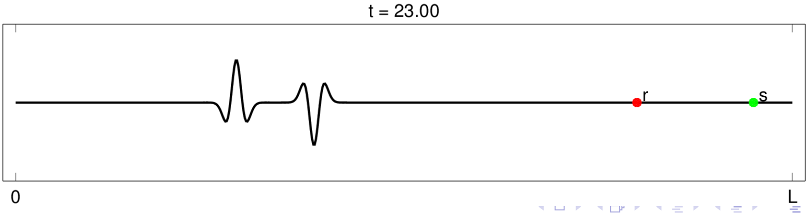

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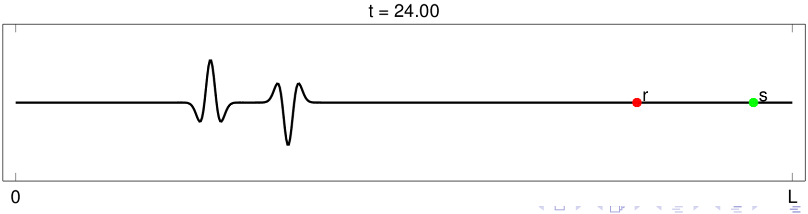

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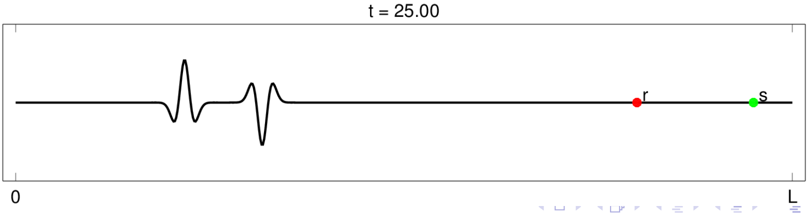

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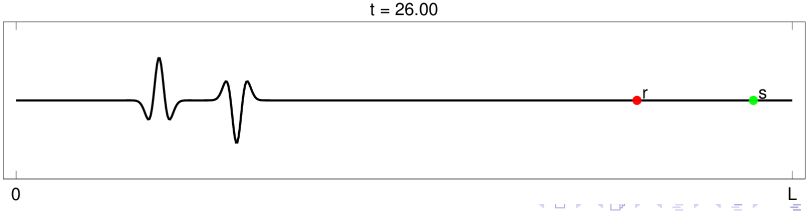

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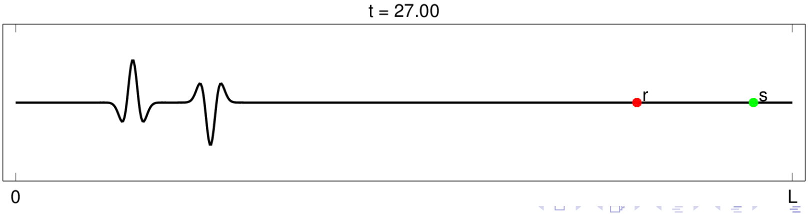

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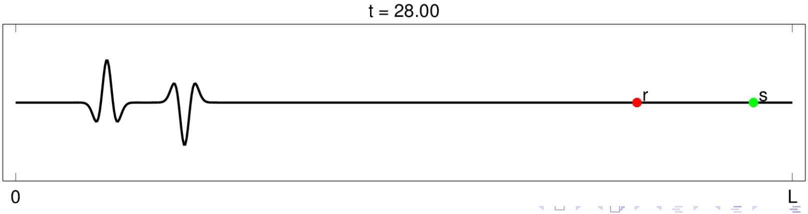

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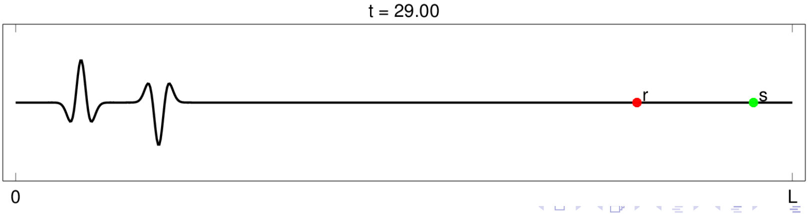

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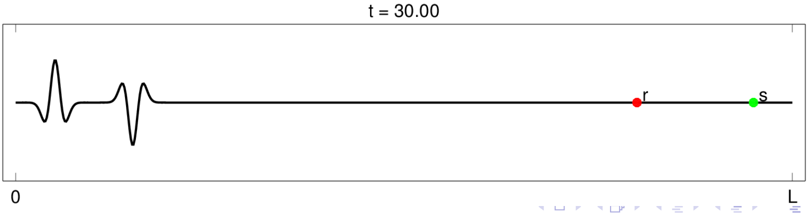

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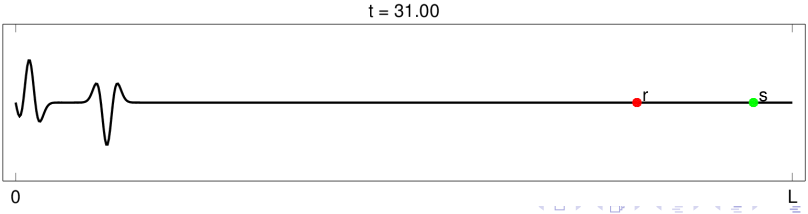

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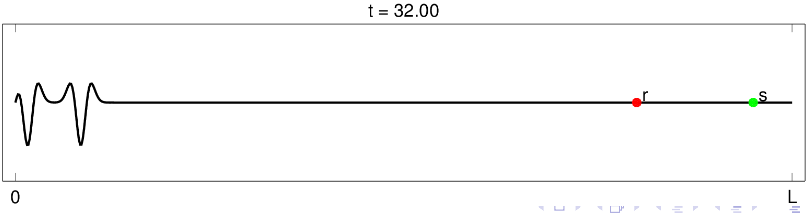

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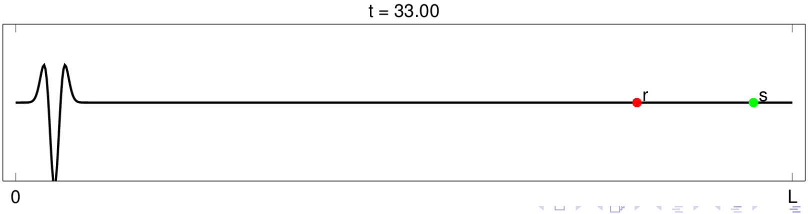

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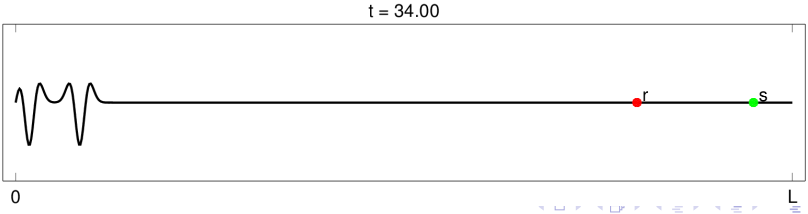

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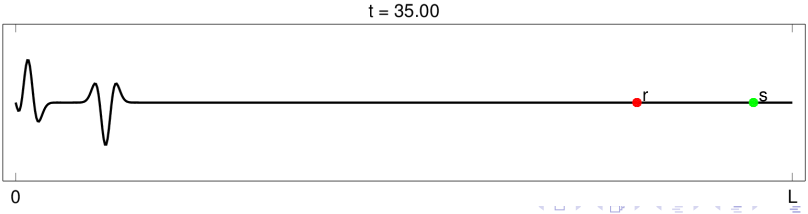

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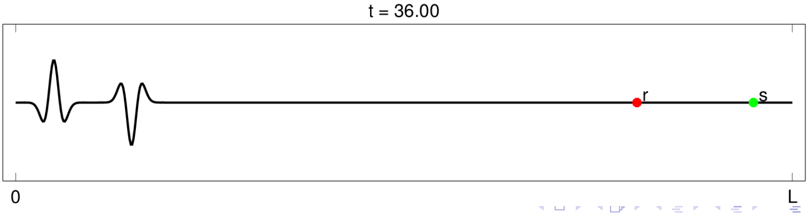

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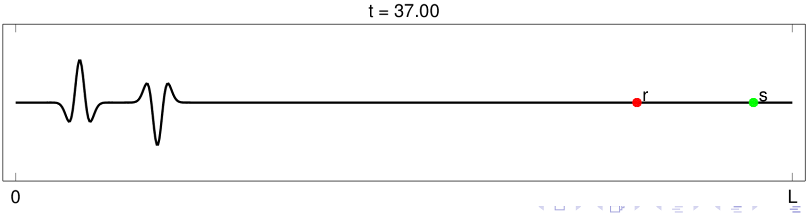

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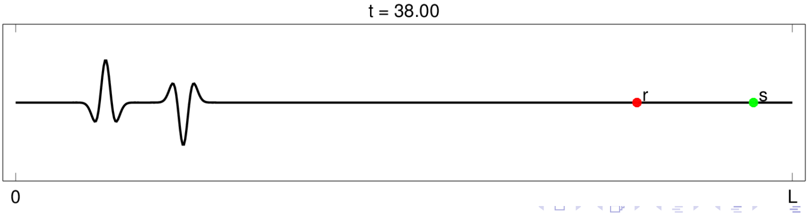

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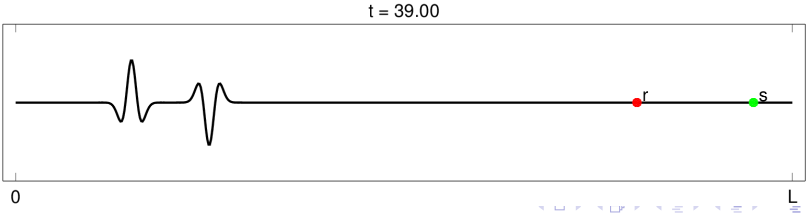

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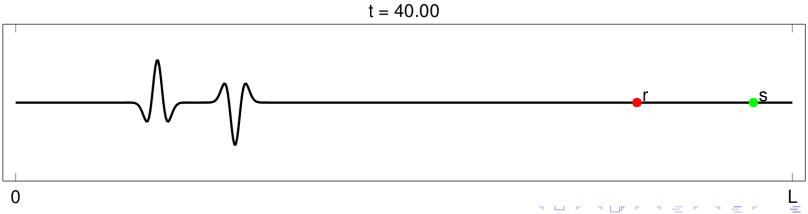

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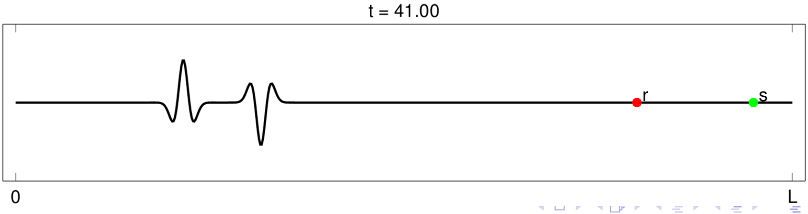

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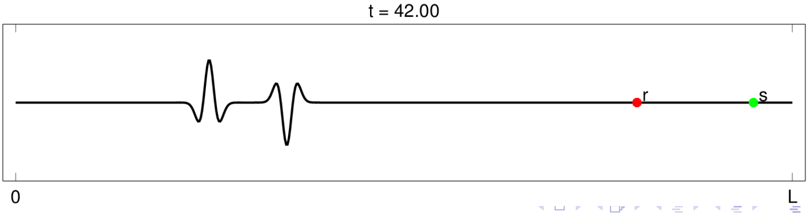

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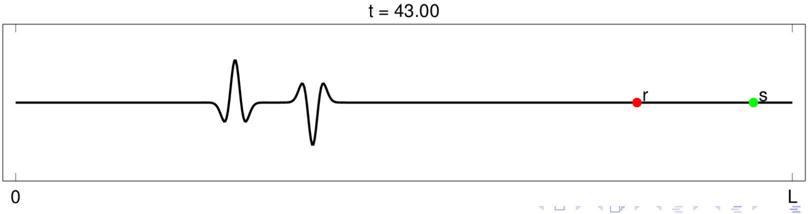

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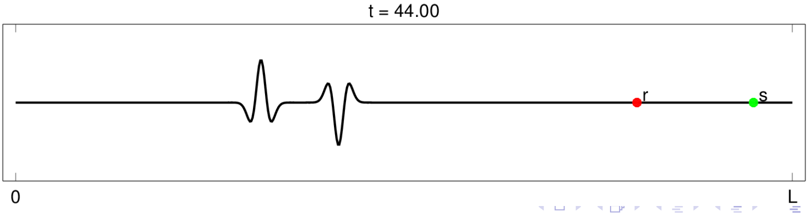

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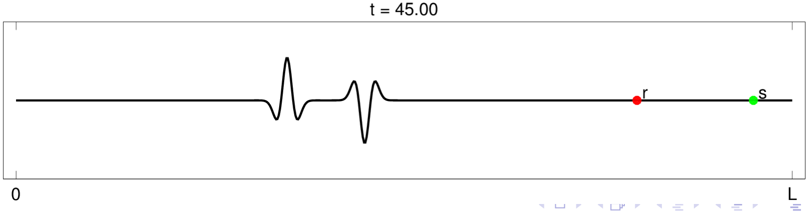

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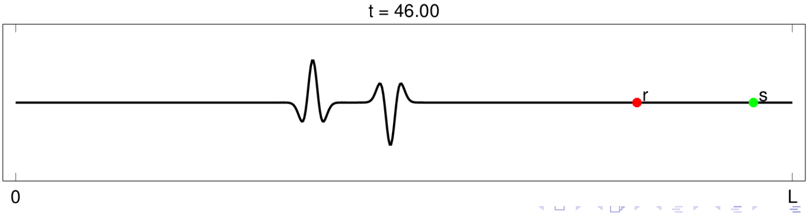

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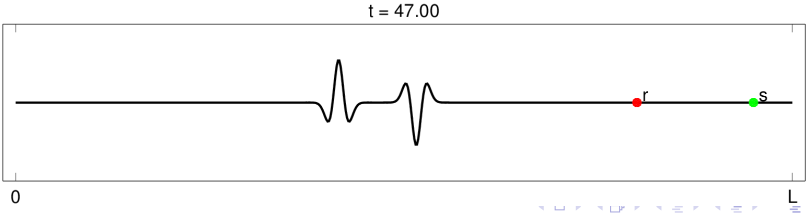

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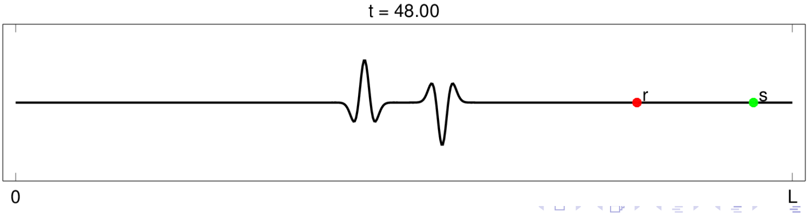

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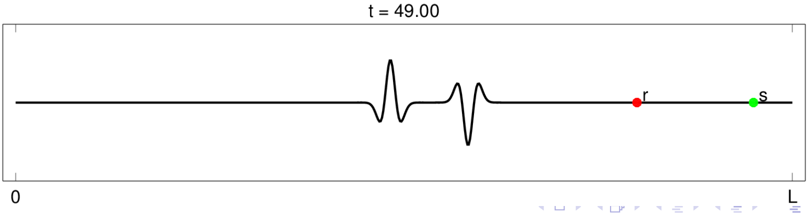

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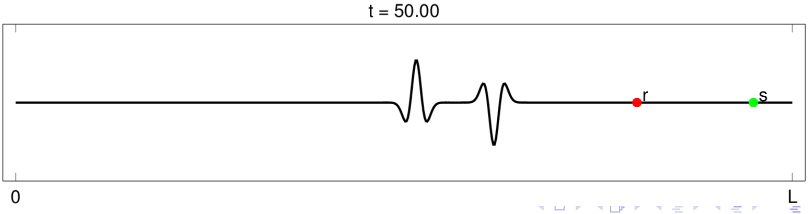

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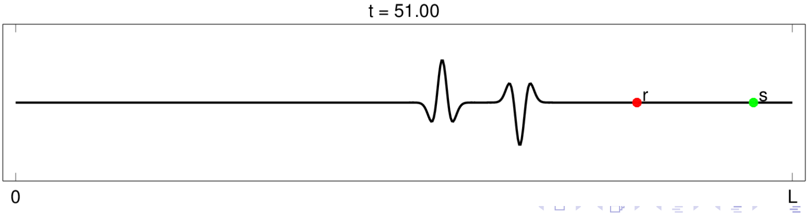

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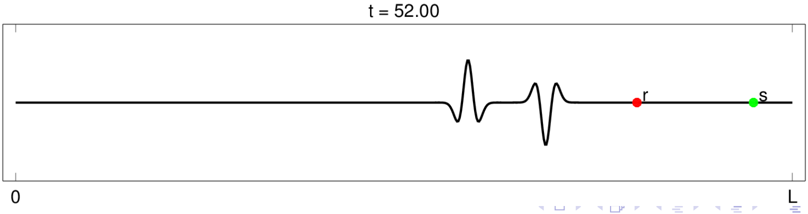

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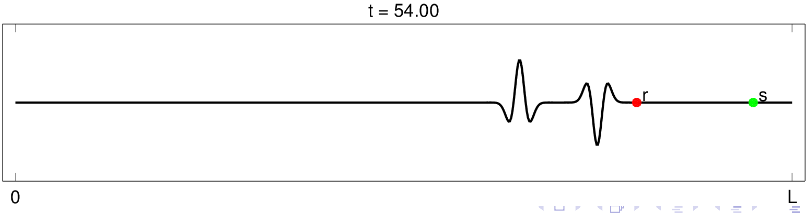

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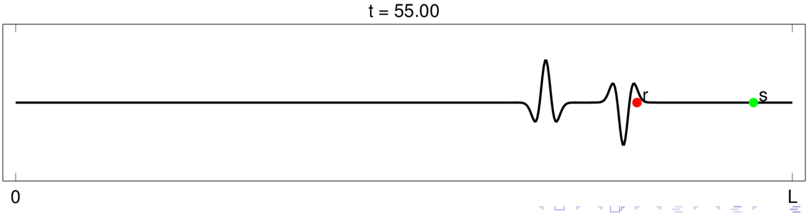

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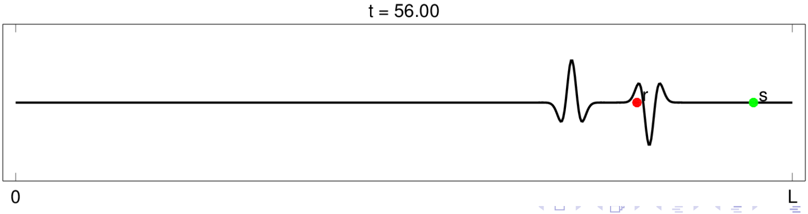

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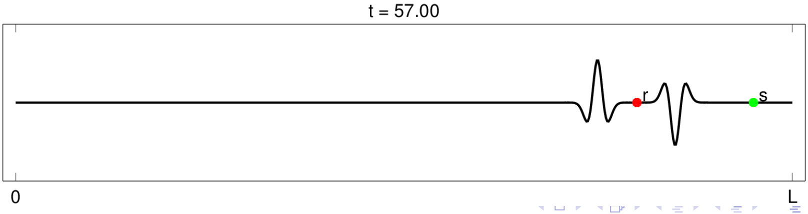

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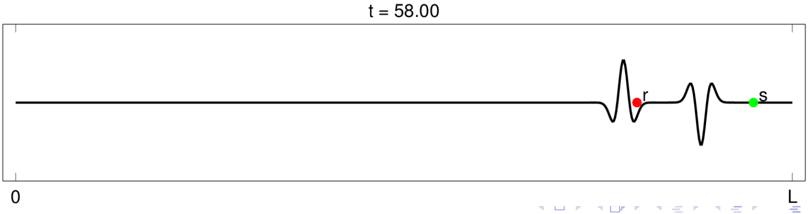

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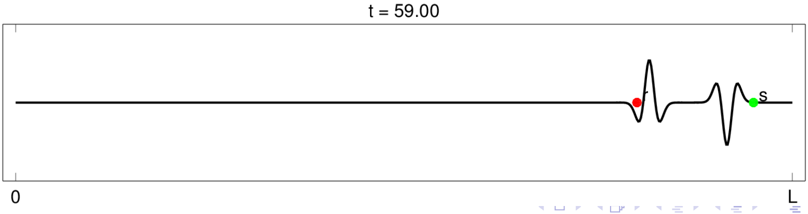

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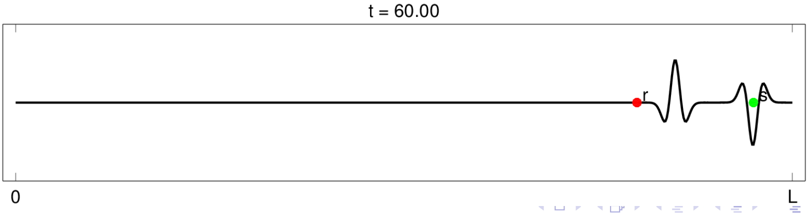

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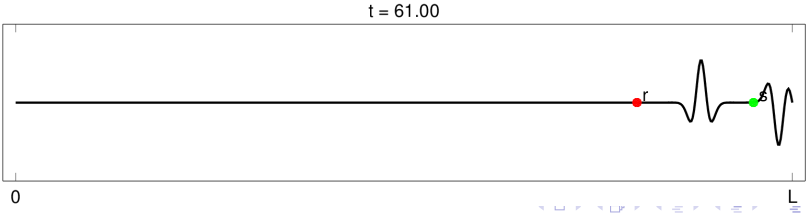

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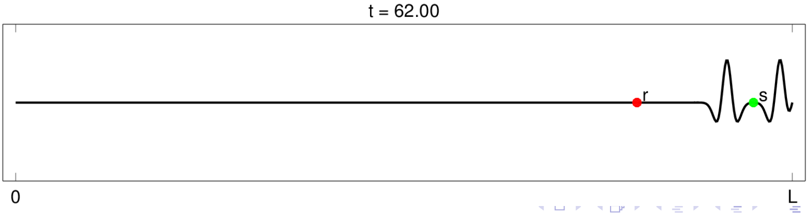

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- The domain contains one source at x_s and one receiver at x_r
- Simulated Numerically using a finite element method
- Wave propagation model

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = f(t)\delta(x - x_s)$$

- Homogeneous Dirichlet boundary conditions and zero initial conditions
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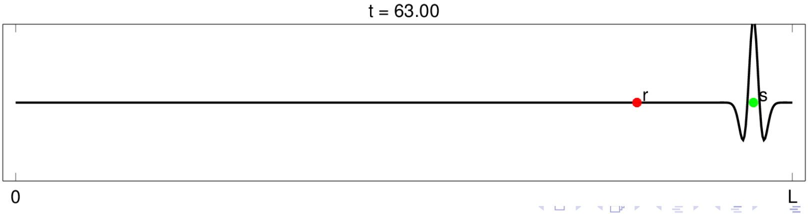

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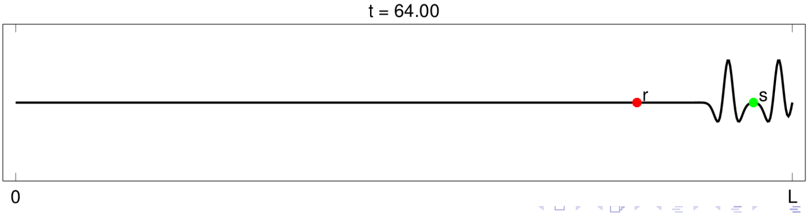

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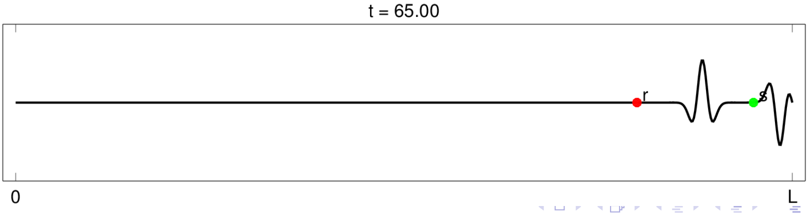

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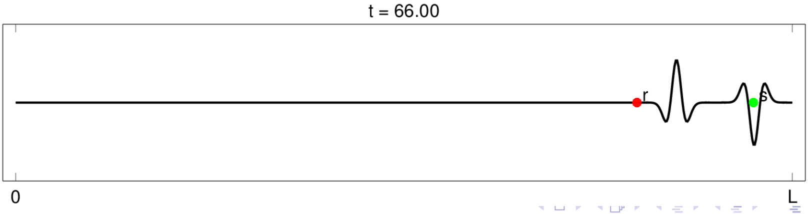

Time domain solution - TR Backward step

- Always performed numerically
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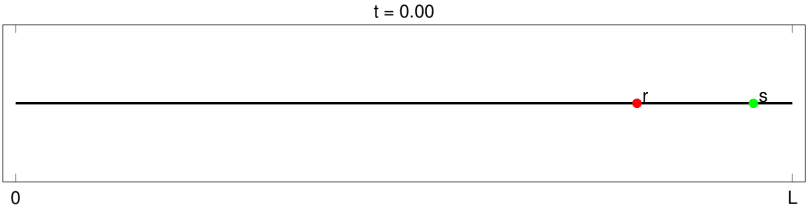
- Example with $t_0 = 3.00$ and total time $T = 66.00$



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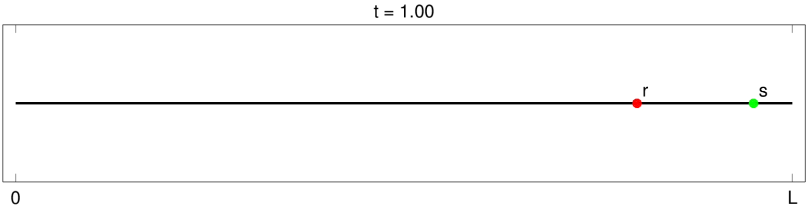

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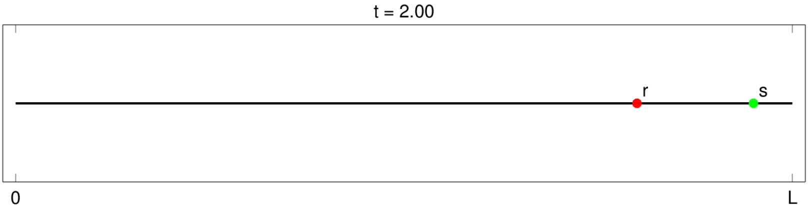

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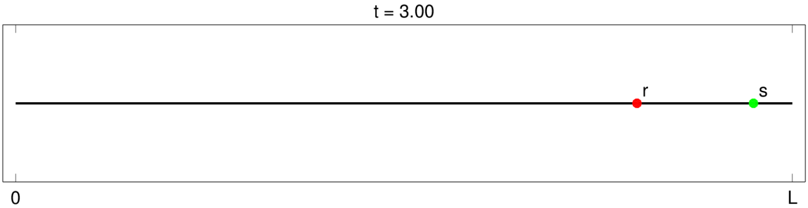

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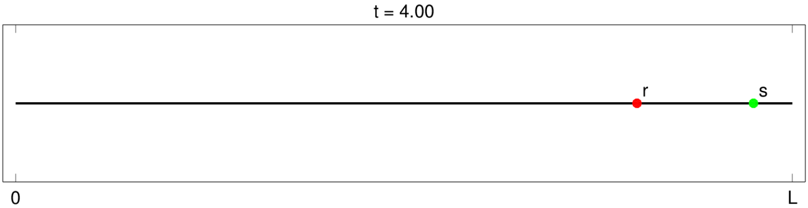

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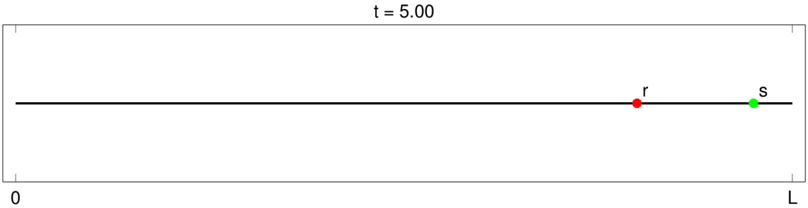

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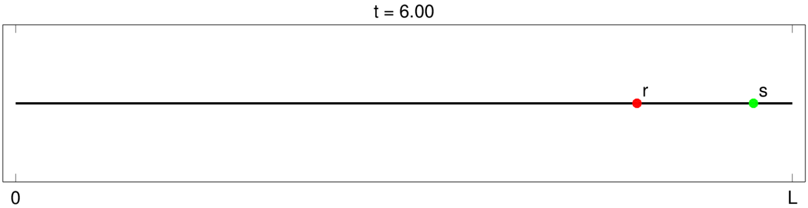

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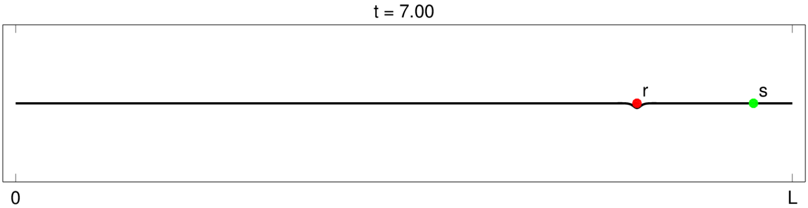

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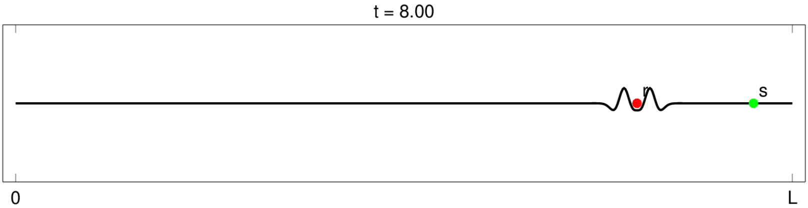

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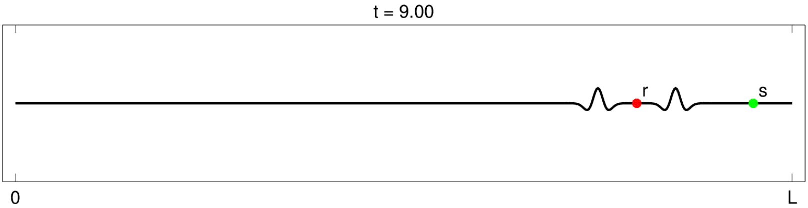

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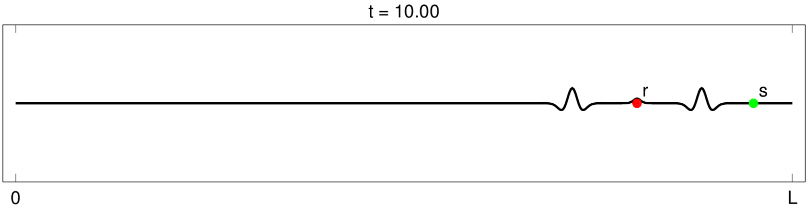

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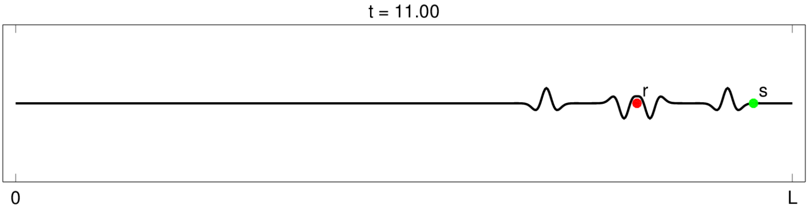

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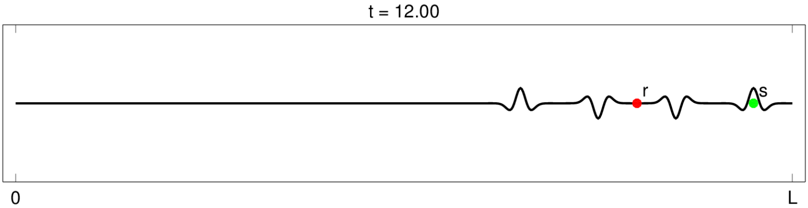

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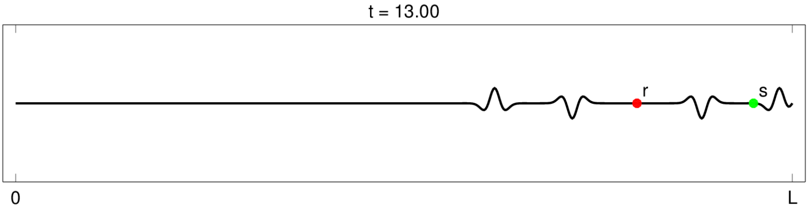

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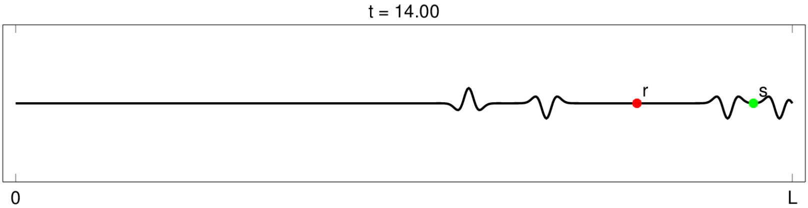

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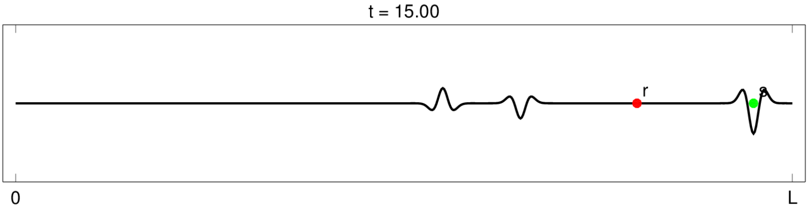

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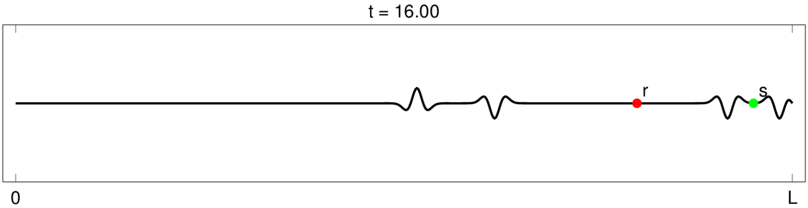

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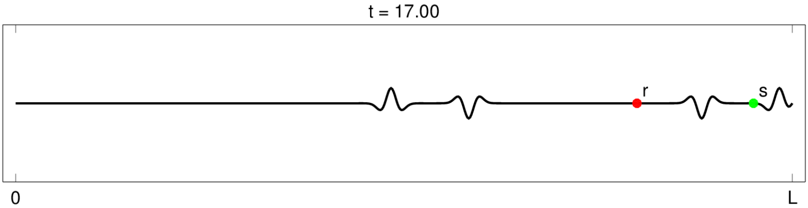

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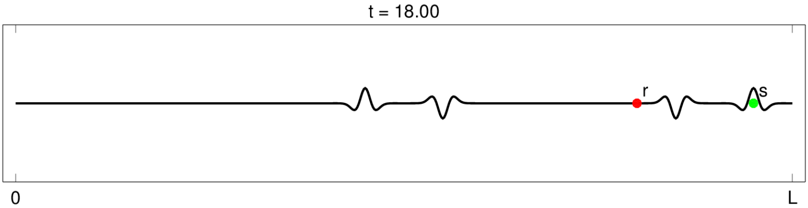

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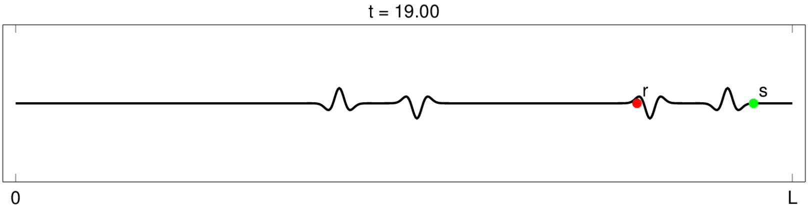

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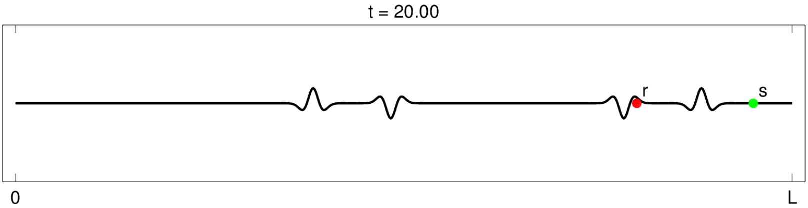

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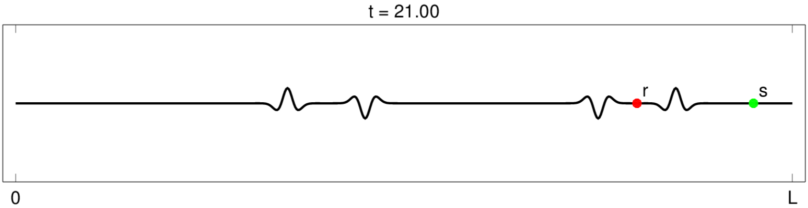

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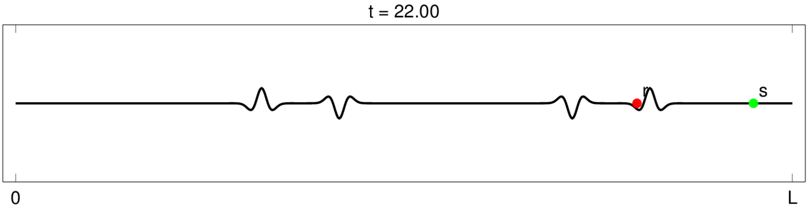

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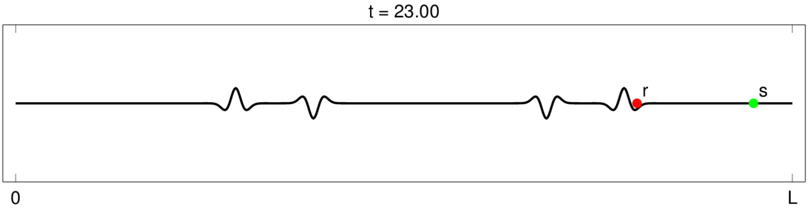

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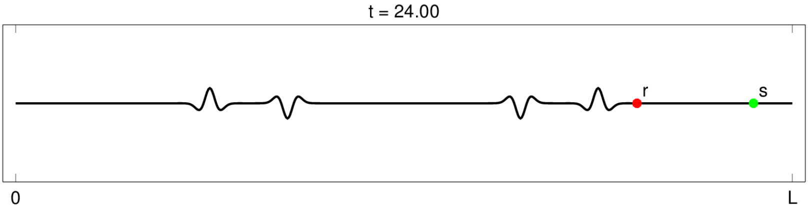

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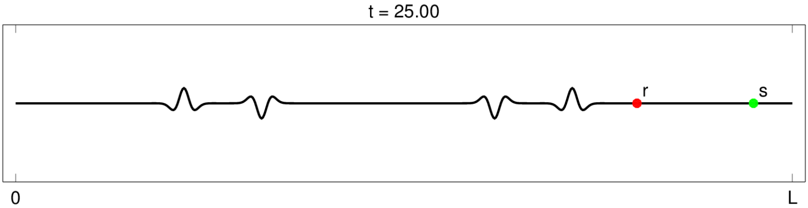

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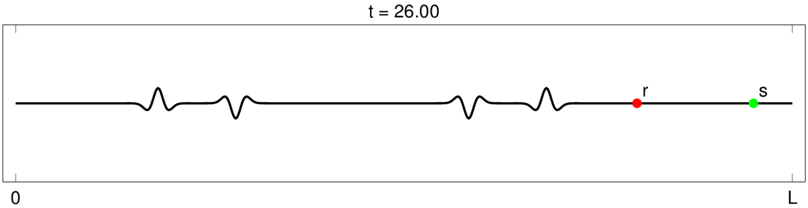

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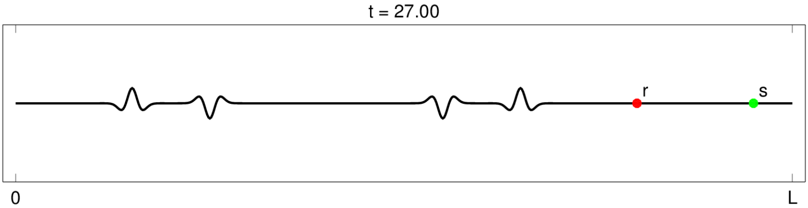

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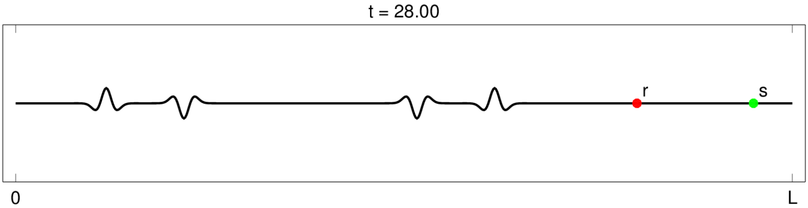

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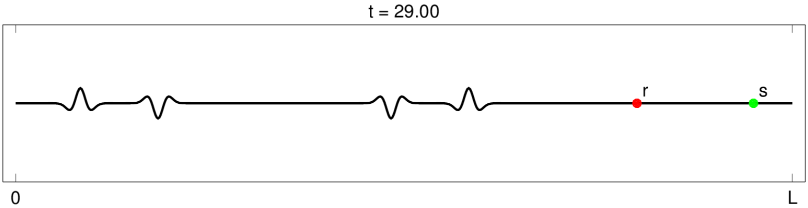

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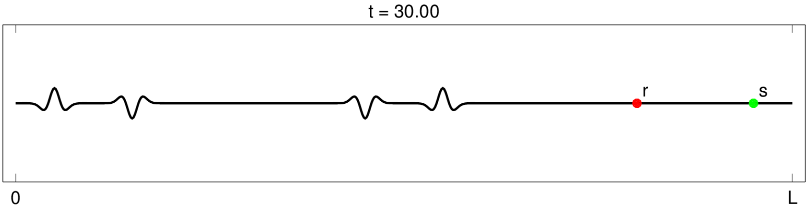

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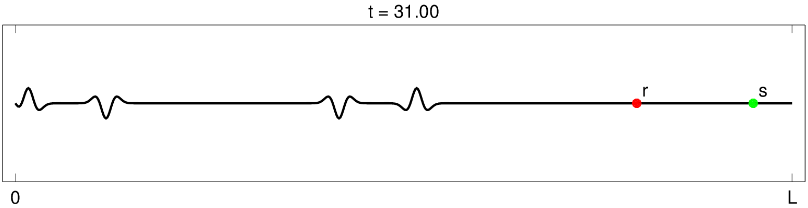

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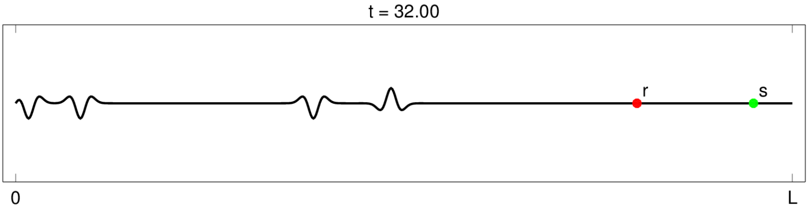

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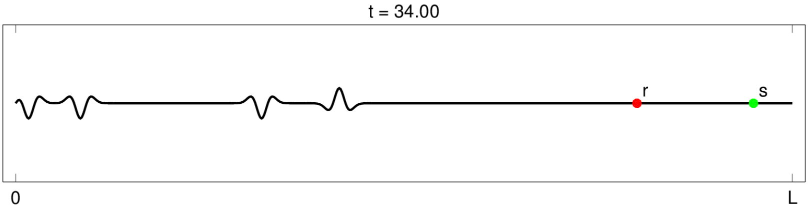

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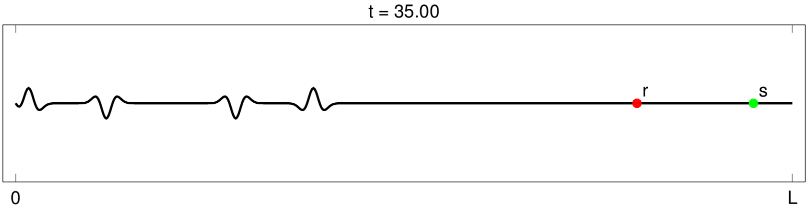

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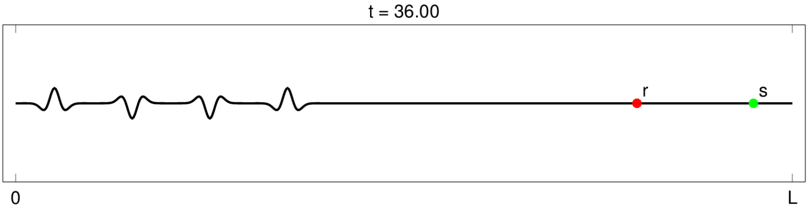

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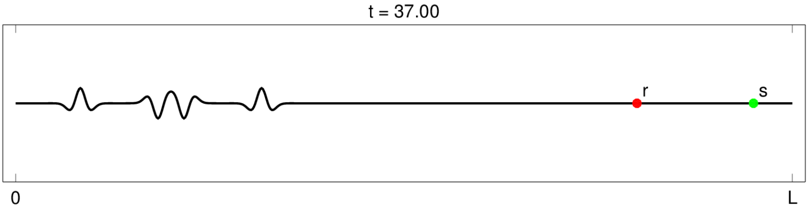

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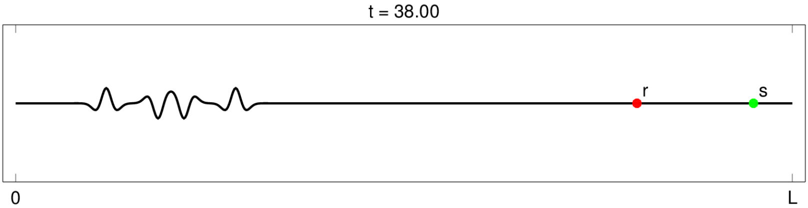

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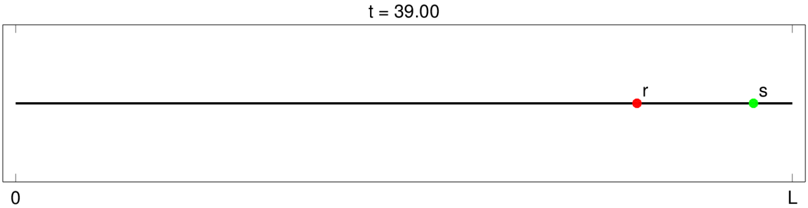

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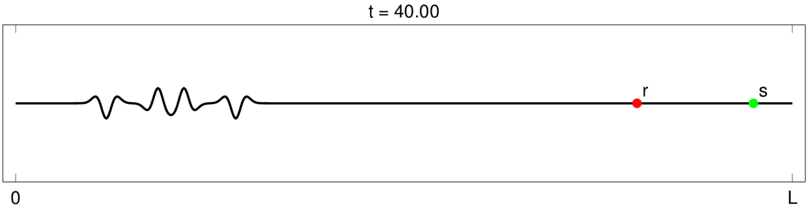

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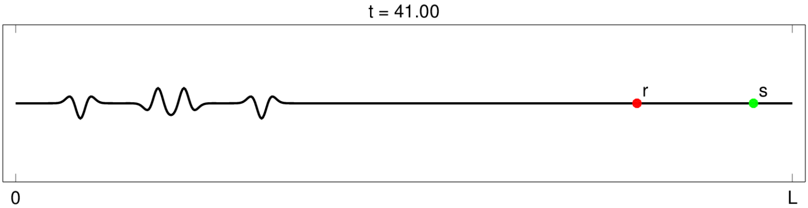

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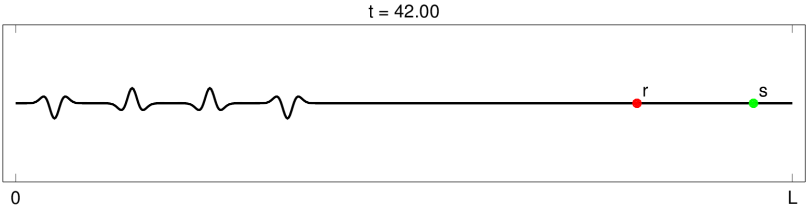

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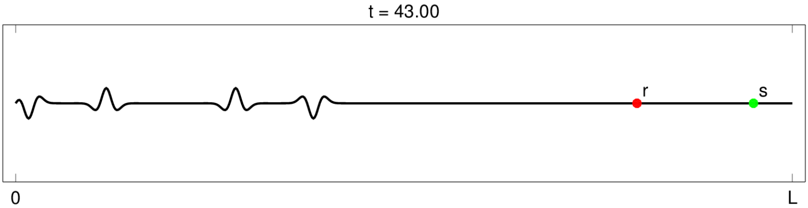

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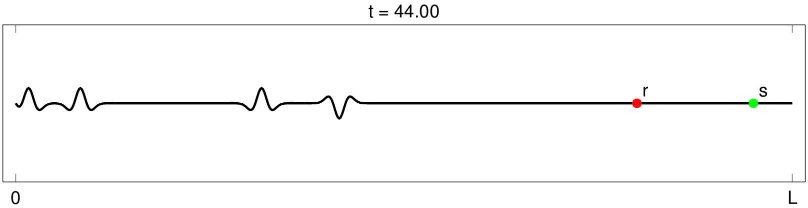

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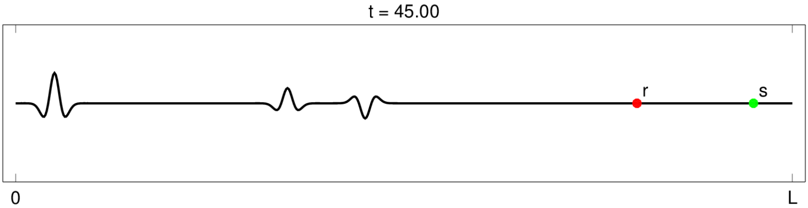

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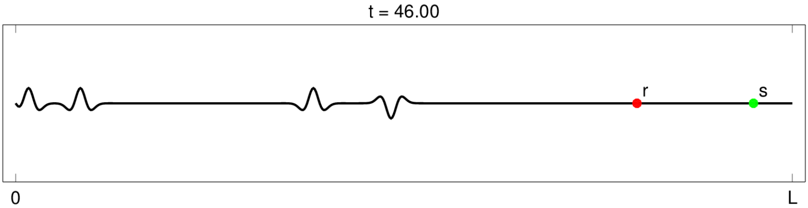

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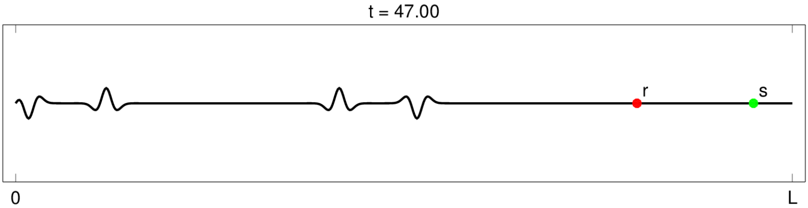

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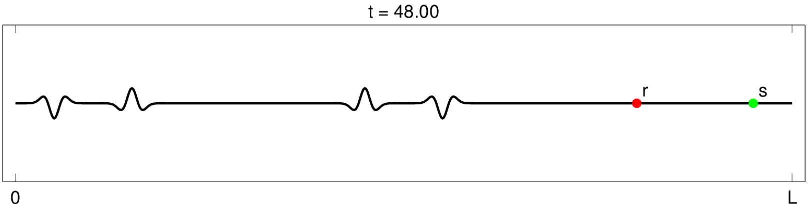

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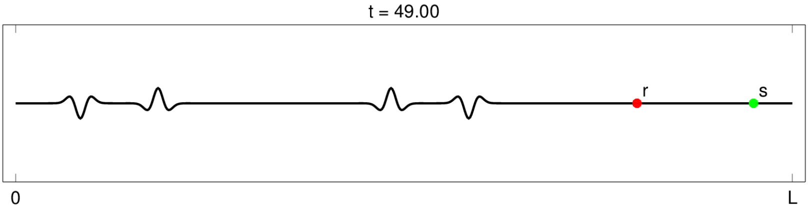

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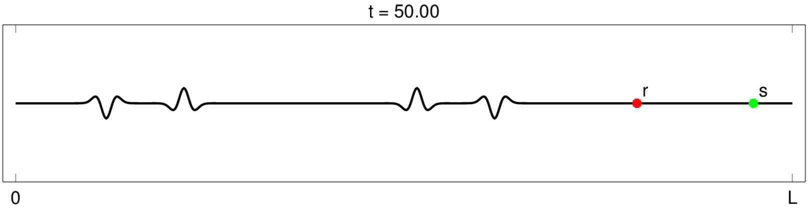

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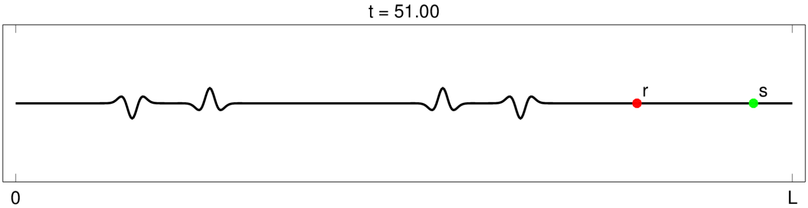

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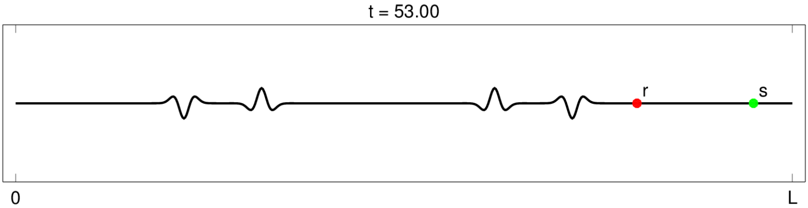

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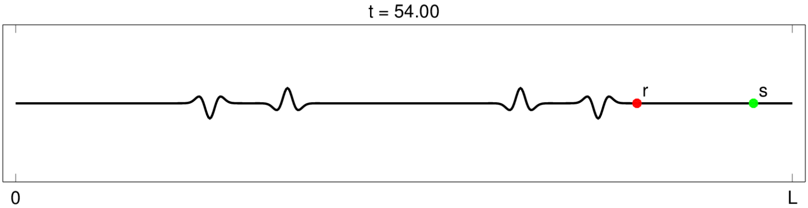

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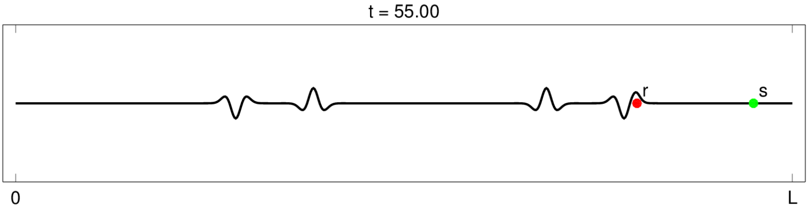

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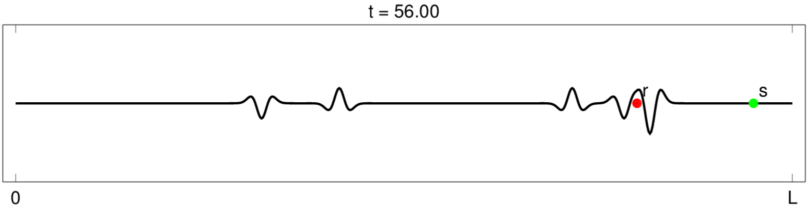

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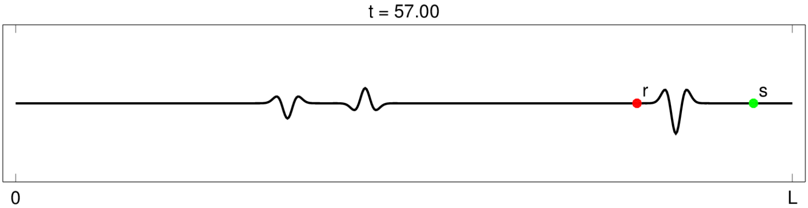

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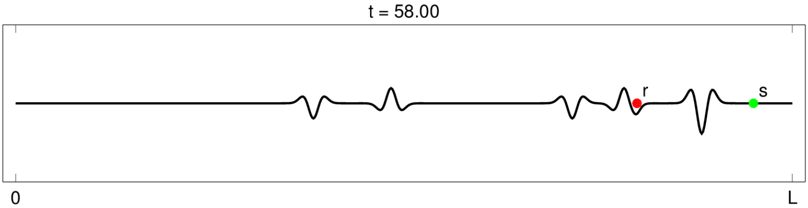

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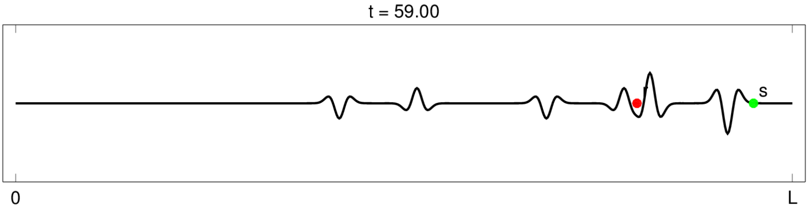

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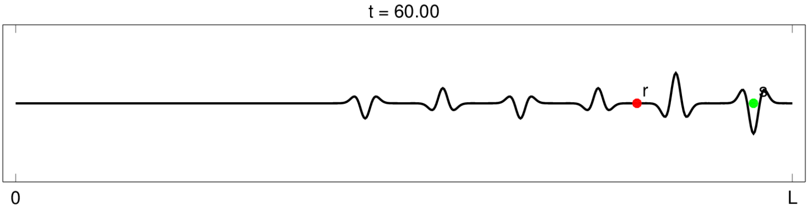
- The recorded signal is time reversed and retransmitted at x_r

$$\frac{1}{c^2}\frac{\partial^2 p^{TR}}{\partial t^2} - \frac{\partial^2 p^{TR}}{\partial x^2} = p(x_r, T-t; x_s)\delta(x - x_r)$$

- Homogeneous Dirichlet boundary conditions and zero initial conditions
- Refocusing at time $t_{RF} = T - t_0$
- Example with $t_0 = 3.00$ and total time $T = 66.00$



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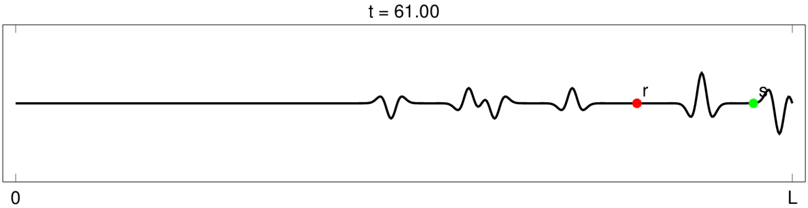

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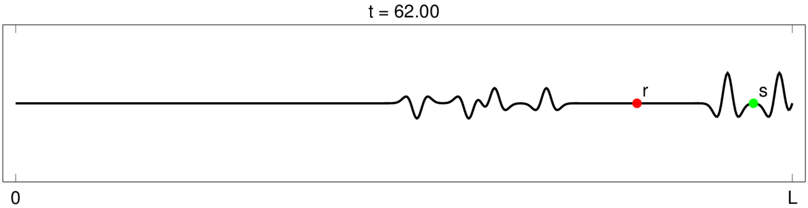

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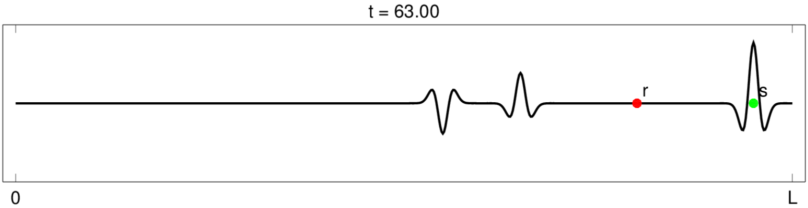

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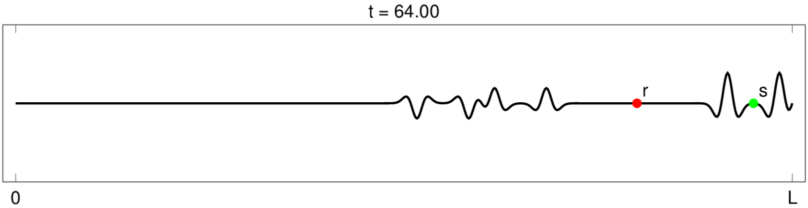

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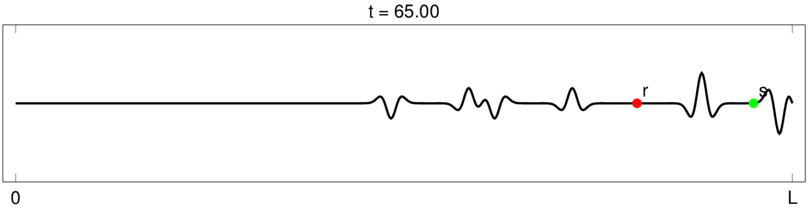

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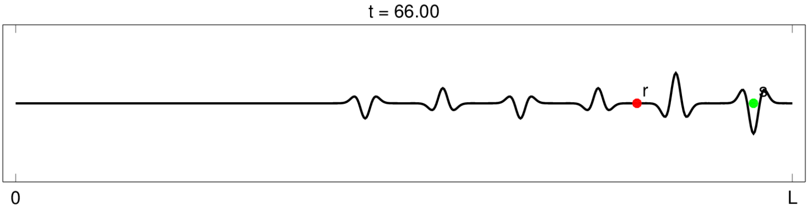

Frequency domain solution - Imaging

- Solution of the backward problem

$$F(x_r, t) = p(x_r, T - t) \Leftrightarrow \hat{F}(x_r, \omega) = \overline{\hat{p}(x_r, \omega)} e^{i\omega T}$$

Frequency domain solution - Imaging

- Solution of the backward problem

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- Evaluation of $p^{TR}(x,t)$ at the refocusing time $T - t_0$

$$p^{TR}(x, t = T - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{p}(x_r,\omega)} \hat{G}(x_r,x,\omega) e^{i\omega t_0} \mathrm{d}\omega$$

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- Imaging functional - numerical approximation

$$\boxed{I(x)=\frac{1}{2\pi}\sum_i\overline{\hat{p}(x_r,\omega_i)}\hat{G}^h(x_r,x,\omega)\Delta\omega_i}$$

Modal expansion

- Data at the receiver

$$p(x_r, t) = f(t) \star_t G(x_s, x_r, t) \Leftrightarrow \hat{p}(x_r, \omega) = \hat{f}(\omega)\hat{G}(x_s, x_r, \omega)$$

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- Substitute in the Imaging functional

$$I(x) = \frac{1}{2\pi}\sum_i |\hat{f}(\omega_i)|^2 \overline{\hat{G}(x_s,x_r,\omega_i)}\hat{G}(x_r,x,\omega)\Delta\omega_i$$

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- Modal expansion formula of the Green's function

$$G^{modal}(x,\xi,\omega) = \sum_{n=1}^{N} \frac{1}{\frac{\omega^2}{c^2} - \lambda_n}\Phi_n(x)\Phi_n(\xi)$$

Modal expansion

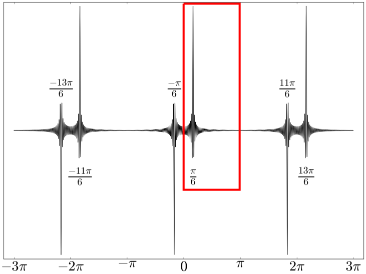
- After the calculations and omitting $\hat{f}(\omega)$

$$\tilde{\mathrm{I}}(x) = C_0 \sum_{i=1}^{3} \left[F_i \sum_{n=1}^{N} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi A_i}{L}\right) \right]$$

i	F_i	A_i
1	1.0	x_s
2	0.5	$x_s + 2x_r$
3	0.5	$x_s - 2x_r$

- Each of the series is a periodic (period 2π) that exhibits exactly one peak every half period

$$\sum_{n=1}^{N} \sin(nx) \sin(n\pi/6)$$



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- The imaging functional exhibits exactly three peaks within the interval [0,L]

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Time domain solution - TR

- source, receiver and 1 defect - small area around x_d with different wave velocity
- each time the original pulse passes from the defect it splits into a transmitted and a reflected component
- Assumption : the incident field p_{inc} is known (the response at the healthy domain)
- scattered field $p_{scat} = p_{tot} - p_{inc}$ to minimize the influence of the source
- The defect acts as a multiple in time source
- p_{scat} is time reversed and retransmitted
- not only one refocusing time but the strongest at $t^{RF} = T - t_1 - t_0$

Frequency domain solution - Imaging

- Data at the receiver - Born approximation

$$\hat{p}_{scat}(x_r, \omega) = k^2 \hat{f}(\omega) \rho \hat{G}(x_s, x_d, \omega) \hat{G}(x_d, x_r, \omega) \quad (1)$$

- It seems natural to define an imaging functional as

$$I(x) = \sum_i \overline{\hat{p}_{scat}(x_r, \omega)} \hat{G}^h(x_r, x, \omega) \hat{G}^h(x, x_s, \omega) \quad (2)$$

- The appearance of the two Green's functions differentiates imaging from TR
- results from the two methods are not comparable unlike the source localization

Modal expansion

- Substituting $\hat{G}^h$ and $\hat{p}_{scat}$ into the Born approximation

$$I(x) = \sum_{\omega} k^2 \rho \left(\hat{f}^h(\omega)\right)^2 \overline{\hat{f}(\omega)\hat{G}(x_s, x_d, \omega)\hat{G}(x_d, x_r, \omega)}\hat{G}(x_r, x, \omega)\hat{G}(x, x_s, \omega)$$

- Using the modal expansion of $\hat{G}$

$$\tilde{\mathrm{I}}(x) = C_1 \left\{ \sum_{i=1}^{13} \left[F_i \sum_{n=1}^{N} \cos\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{2n\pi A_i}{L}\right) \right] + \sum_{n=1}^{N} \cos\left(\frac{2n\pi x}{L}\right) \right\} + C_2$$

Modal expansion

$$\tilde{\text{I}}(x) = C_1 \left\{ \sum_{i=1}^{13} \left[F_i \sum_{n=1}^{N} \cos\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{2n\pi A_i}{L}\right) \right] + \sum_{n=1}^{N} \cos\left(\frac{2n\pi x}{L}\right) \right\} + C_2$$

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1	1.0	x_d	4	0.5	$x_d - x_s$	7	0.5	$x_d + x_r$	10	0.25	$x_d - x_s - x_r$
2	1.0	x_s	5	0.5	$x_d + x_s$	8	0.5	$x_s - x_r$	11	0.25	$x_d - x_s + x_r$
3	1.0	x_r	6	0.5	$x_d - x_r$	9	0.5	$x_s + x_r$	12	0.25	$x_d + x_s - x_r$
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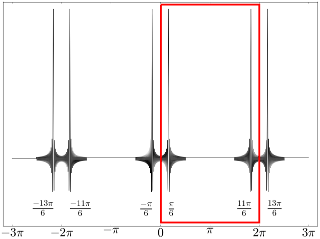
$$\sum_{n=1}^{N} \cos(nx)\cos(n\pi/6)$$



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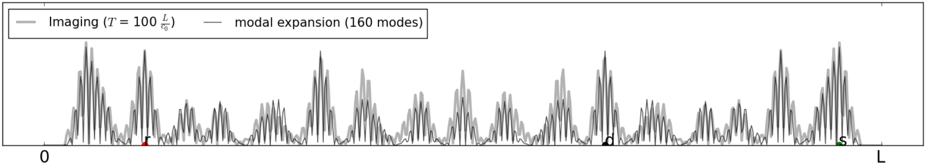
- The imaging functional exhibits exactly 26 peaks within the interval [0,L]



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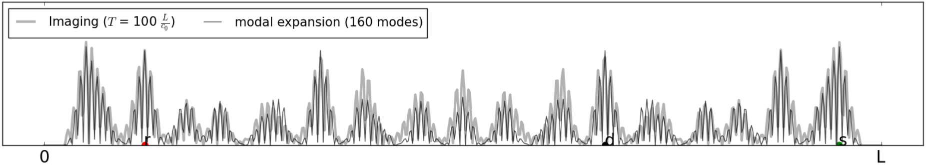

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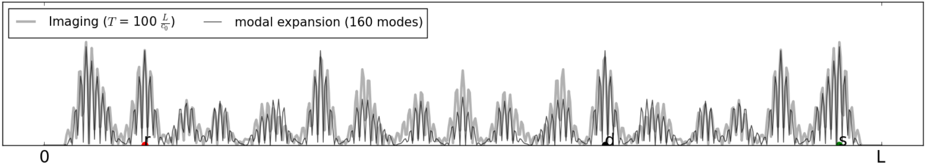

Total experiment time

- In defect localization the choice of T is of significant importance
- Time reversal
  - If the wave travels many times across the domain, p_{scat} becomes complicated
  - Best results for $T = \frac{|x_s - x_d|}{c_{ref}} + \frac{|x_d - x_r|}{c_{ref}} + 2t_0$
  - Because x_d in not known, optimum choice $T = \frac{2L}{c_{ref}} + 2t_0$
- Imaging
  - The data at the receiver and Green's functions are calculated in the time domain and then FT
  - As a result T can be taken into account similarly to the TR case
- Modal expansion
  - It is assumed here that $T = \infty$ and thus no further discussion is meaningful
  - If Imaging is performed for very large $T(\rightarrow \infty)$ it approaches the modal expansion

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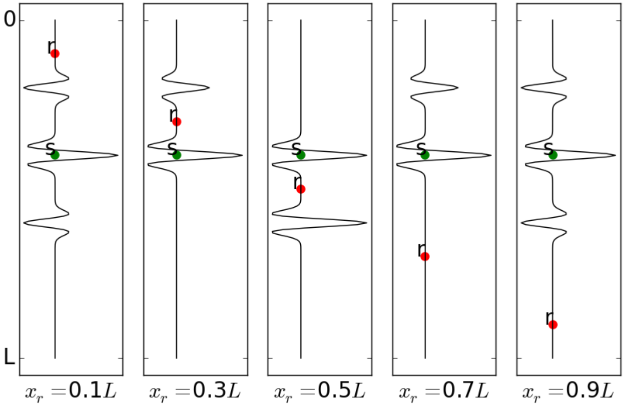
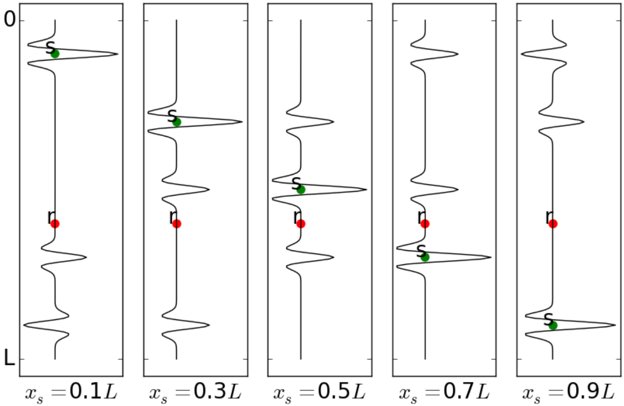
Source localization - Imaging

- Investigation of the receiver position



Source localization - Imaging

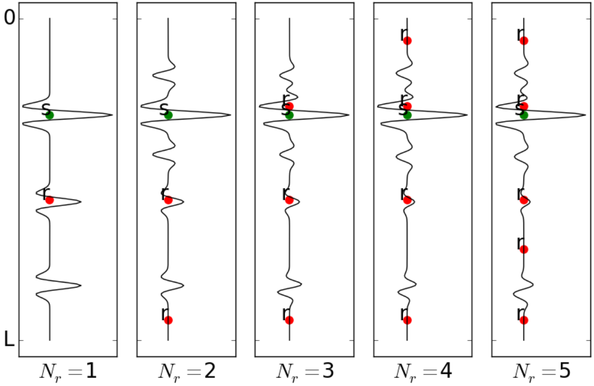
- Investigation of the source position



Source localization - Imaging

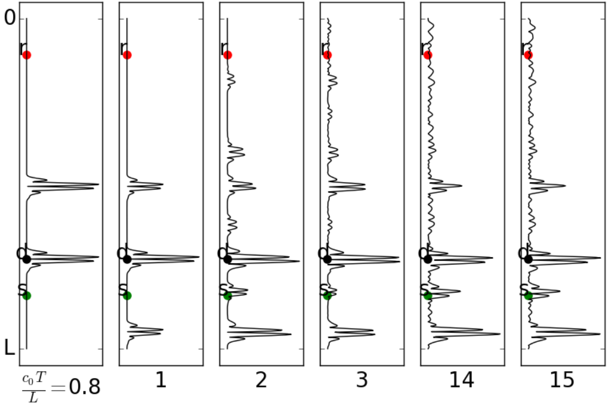
- Improvement of the SNR by increasing the number of receivers
- Linear relationship between SNR and N_r

$$\mathrm{I}(x) = \sum_{\omega} \sum_{r=1}^{N_r} \overline{\widehat{p}(x_r, \omega)} \widehat{G}^h(x_r, x, \omega). \qquad (3)$$



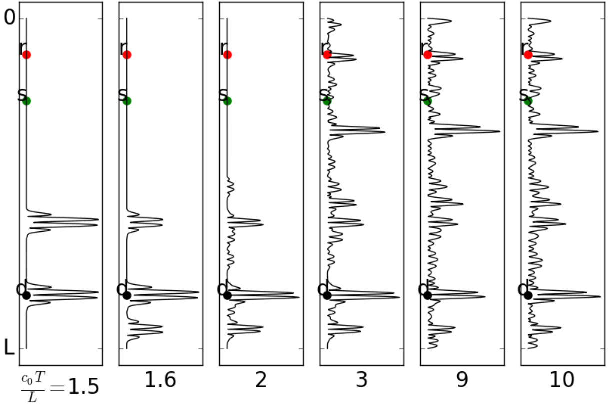
Defect localization - TR

- Time reversal for defect localization example 1
- Investigation of total experiment time T



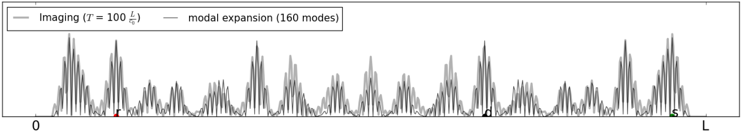
Defect localization - TR

- Time reversal for defect localization example 2
- Investigation of total experiment time T

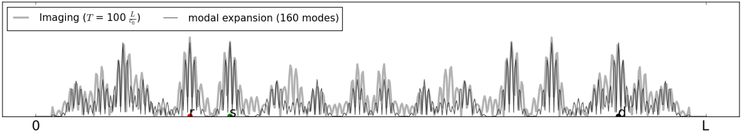


- Comparison of Imaging (T=∞) and modal expansion

- Example 1

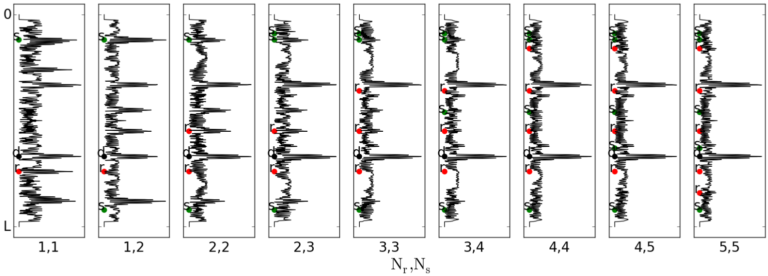


- Example 2



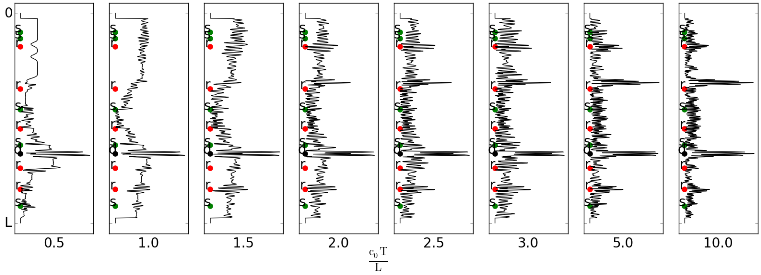
- Improvement of the image quality using higher number of sources and receivers

$$\mathrm{I}(x) = \sum_{\omega} \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} \overline{\widehat{p}_{scat}(x_r, \omega)} \widehat{G}^h(x_r, x, \omega) \widehat{G}^h(x, x_s, \omega) \quad (4)$$



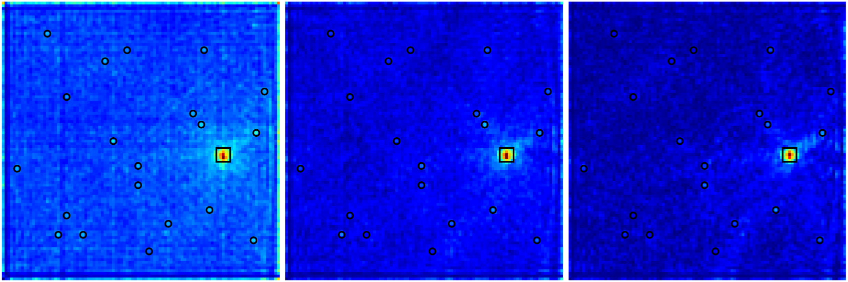
Defect localization - Imaging

- Improvement of the SNR by reducing total time T



Defect localization 2D example - Imaging

- 2D bounded domain, contains 20 receivers that act as sources as well
- Investigation of the total experiment time
- Much better results compared to the 1D case because :
  - In the 1D case the defect separates the domain in two parts
  - Large number of receivers and sources



$T = 10\frac{\sqrt{2}L}{c_0}$ $T = 4\frac{\sqrt{2}L}{c_0}$ $T = 2\frac{\sqrt{2}L}{c_0}$

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Summary and Conclusions

- TR and Imaging for damage localization in 1D acoustic bounded media
- Exploitation of the similarities and disparities between source and defect localization problems
- Analyzed the noise resulting from the presence of the boundaries
- Investigated the effect of the total experiment time T
- Proposition of techniques for the improvement of SNR and image quality
- Results from defect localization in 2D domain are very promising

Future work

- Extend the Imaging techniques in elastic media and higher dimensions
- Investigate the localization process for noisy recordings
- Utilize passive only recordings due to ambient vibration
- Apply techniques for the separate localization of multiple defects [5]
- Apply this methodology on structures with complex geometry [6] for the development of SHM systems

5. CG Panagiotopoulos, Y Petromichelakis, C Tsogka (2015) Time Reversal in elastodynamics and applications to Structural Health Monitoring, COMPDYN 2015

6. CG Panagiotopoulos, Y Petromichelakis, C Tsogka (2015) Time reversal and imaging for structures, Chapter in "Dynamic Response of Infrastructure to Environmentally Induced Loads"